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Potent divisors of the characteristic matrix of a minimum simple square ante-slope.[*]

By

NRIPENDRANATH GHOSH

[*Read February 29th, 1920*]

The most general simple square ante-slopes of order m are of two types, the square matrix (1) representing the first type and the square matrix (2) representing the second type.

$$(1)\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ 0 & a_{22} & a_{23} & \cdots & a_{2m} \\ 0 & 0 & a_{33} & \cdots & a_{3m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{mm} \end{bmatrix},\quad (2)\ \begin{bmatrix} a_{mm} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{3m} & \cdots & a_{33} & 0 & 0 \\ a_{2m} & \cdots & a_{23} & a_{22} & 0 \\ a_{1m} & \cdots & a_{13} & a_{12} & a_{11} \end{bmatrix}.$$

Since either of the two types can be converted into the other by a symmetric derangement, it will be sufficient for our purpose to consider simple square ante-slopes of the first type. When minimum such a slope contains not more than one non-zero element in each horizontal row and not more than one non-zero element in each vertical row.

Let $A = [a]_m^m$ be a minimum simple square ante-slope of the first type. We shall find the potent divisors of its characteristic matrix or simply the characteristic potent divisors of A.

It will be proved that A can always be converted by a symmetric derangement into a compartite matrix in standard form. The successive parts of the compartite matrix can be found by grouping the diagonal elements of A in the way which will now be described.

First, put aside every non-zero element of the parametric diagonal. Let there be u such elements forming the group

$$a_{h_1 h_1},\ a_{h_2 h_2},\ \cdots\ a_{h_v h_u}, \qquad \ldots\ \text{(H)}$$

where

$$[h_1 h_2 \ldots h_u] \qquad \ldots\ (h)$$

is a corranged minor of $[1\ 2 \ldots m]$.

---

[*] Communicated by Prof. C. E. Cullis, M.A., Ph.D.

Next, put aside every zero-element of the parametric diagonal, which is such that neither the horizontal row nor the vertical row through it contains a non-zero element. Let there be x such elements forming the group

$$a_{k_1k_1},\ a_{k_2k_2},\ \dots\ a_{k_vk_v}. \qquad \dots \text{(K)}$$

where

$$[\,k_1k_2\dots k_v\,] \qquad \dots (k)$$

is a corranged minor of $[1, 2\dots m]$.

Now consider the remaining zero elements of the parametric diagonal each of which is such that either the horizontal row or the vertical row through it contains a non-zero element. Let the first of these elements be $a_{p_1p_1}$. Then there is a non-zero element in the p_1th horizontal row of A. Let this be $a_{p_1p_2}$, where $p_2>p_1$. Draw straight lines joining the zero element $a_{p_1p_1}$ to $a_{p_1p_2}$ and $a_{p_1p_2}$ to the zero element $a_{p_2p_2}$. If there is any non-zero element in the p_1th horizontal row of A, let this be $a_{p_2p_3}$, where $p_3>p_2$. Draw straight lines joining the zero element $a_{p_2p_2}$ to $a_{p_2p_3}$ and $a_{p_2p_3}$ to the zero element $a_{p_3p_3}$. If there is any non-zero element in the p_3th horizontal row of A proceed as before, otherwise stop. When this process has been continued as long as possible, let the zero elements of the parametric diagonal which have been used be i in number forming the group

$$a_{p_1p_1},\ a_{p_2p_2},\ \dots\ a_{p_ip_i}. \qquad \dots \text{(P)}$$

where

$$[\,p_1p_2\dots p_i\,] \qquad \dots (p)$$

is a corranged minor of $[1{\cdot}2\dots m]$ having no element in common with either of the sequence (h) and (k).

If there are any remaining elements of the parametric diagonal not included in the groups (H), (K), (P), let the first of these be the zero element $a_{q_1q_1}$. Then there is a non-zero element in the q_1th horizontal row of A. Let this be $a_{q_1q_2}$, when $q_2>q_1$. Draw straight lines joining the zero element $a_{q_1q_1}$ to $a_{q_1q_2}$ and $a_{q_1q_2}$ to the zero element $a_{q_2q_2}$. If there is any non-zero element in the q_2th horizontal row of A, let this be $a_{q_2q_3}$, where $q_3>q_2$. Draw straight lines joining the zero element $a_{q_2q_2}$ to $a_{q_2q_3}$, and $a_{q_2q_3}$ to the zero element $a_{q_3q_3}$. If there is any non-zero element in the q_3th horizontal row of A proceed

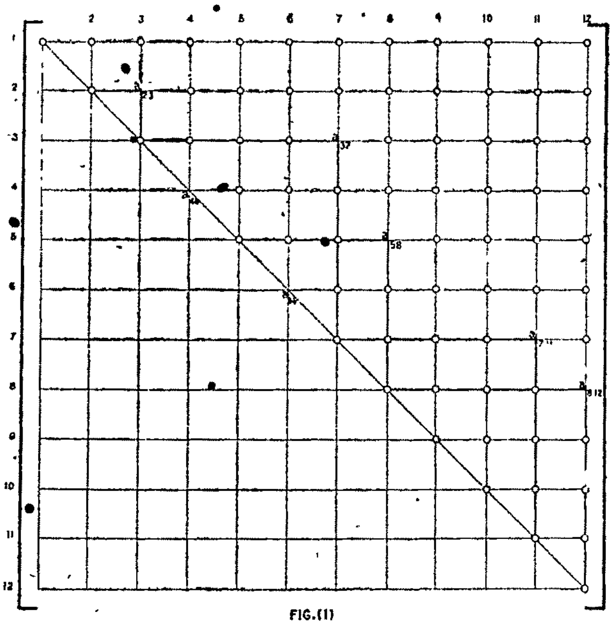
a23
a37
a46
a58
a64
a711
a812

FIG. (1)

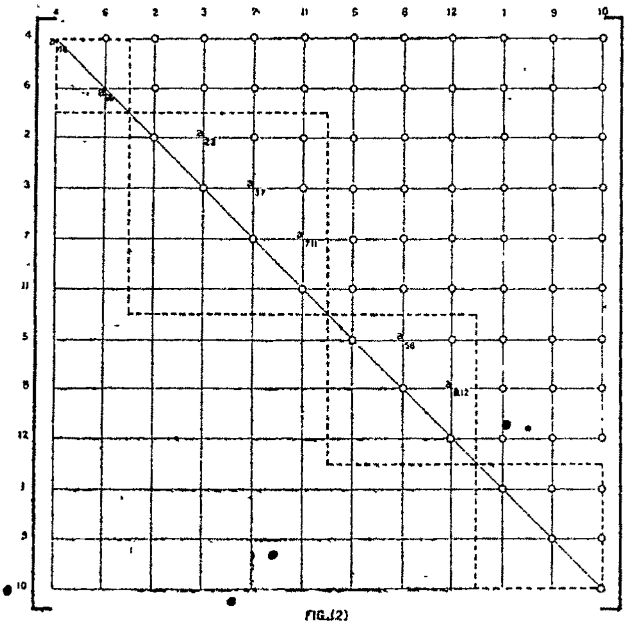
a46
a64
a23
a37
a711
a58
a812

FIG. (2)

as before, otherwise stop. When this process has been continued as long as possible, let the zero elements of the parametric diagonal which have been used be j in number forming the group

$$a_{q_1q_1},\ a_{q_2q_2}\ \dots\ a_{q_jq_j} \qquad \dots \quad (Q)$$

where

$$[\,q_1q_2\dots q_j\,] \qquad \dots \quad (q)$$

is a corranged minor of $[1\ 2\dots m]$ having no element in common with any one of the sequences (h), (k), (p).

In this way we can divide the elements of the parametric diagonal into the groups (h), (k), and a certain number of groups (P), (Q), (R),... all formed in succession in the same way as the group (P).

Then the symmetric minors

$$H_o = \left[\begin{array}{cc} h_1 & h_2\dots h_u \\ & a \\ h_1 & h_2\dots h_u \end{array}\right],\ K_o = \left[\begin{array}{cc} k_1 & k_2\dots k_v \\ & a \\ k_1 & k_2\dots k_v \end{array}\right],\ P_o = \left[\begin{array}{cc} p_1 & p_2\dots p_i \\ & a \\ p_1 & p_2\dots p_i \end{array}\right] \dots,$$

are mutually complementary corranged parts of A. Thus A can be converted by a symmetric derangement into a compartite matrix in standard form whose successive parts are

$$H_o,\ P_o,\ Q_o,\ R_o\dots K_o.$$

In the same way the characteristic matrix $A(x)$ of A can be converted into a compartite matrix in standard form whose successive parts are the characteristic matrices $H_o(x)$, $P_o(x)$, $Q_o(x)\dots K_o(x)$ of $H_o, P_o, Q_o\dots K_o$ respectively; and the potent divisors of $A(x)$ are the potent divisors of these successive parts. Thus $A(x)$ has

u potent divisors $x-a_{h_1h_1},\ x-a_{h_2h_2},\ \dots x-a_{h_uh_u}$ of $H_o(x)$,

v potent divisors each being x ... of $K_o(x)$,

the single potent divisor x^i ... of $P_o(x)$,

the single potent divisor x^j ... of $Q_o(x)$,

and so on.

Consider, for example, the minimum simple square ante-slope of the first type $[a]_{12}^{12}$ (shown in Fig. 1) in which the non-zero elements are

$$a_{23},\ a_{37},\ a_{44},\ a_{53},\ a_{66},\ a_{7,11},\ a_{8,12}.$$

In this case the group H is formed by the non-zero parametric elements

$$a_{44},\ a_{66}\dots;$$

the group K consists of the zero elements

$$a_{11},\ a_{99},\ a_{10,10}\ ;$$

the group P consists of the zero elements

$$a_{22},\ a_{33},\ a_{77},\ a_{11,11}\ ;$$

and the group Q consists of the zero elements

$$a_{55},\ a_{88},\ a_{12,12}\ ;$$

so that the symmetric minors are –

$$H_0=\begin{bmatrix} a_{14} & 0 \\ 0 & a_{66} \end{bmatrix},\ K_0=\begin{bmatrix} 1 & 9 & 10 \\ & a & \\ 1 & 9 & 10 \end{bmatrix},\ P_0=\begin{bmatrix} 0 & a_{23} & 0 & 0 \\ 0 & 0 & a_{37} & 0 \\ 0 & 0 & 0 & a_{7,11} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Q_0=\begin{bmatrix} 0 & a_{58} & 0 \\ 0 & 0 & a_{8,12} \\ 0 & 0 & 0 \end{bmatrix}.$$

Fig. 2 shows the symmetric derangement of Fig. 1. It is a compartite matrix, the 4 successive parts being shown by dotted lines.

The characteristic potent divisors of the first part H_0, are 2 in number.

$$x-a_{44},\ x-a_{66}\ ;$$

the single characteristic potent divisor of the second part P_0 is

$$x^4\ ;$$

the single characteristic potent divisor of the third part Q_0 is

$$x^3\ ;$$

the characteristic potent divisors of the last part, the zero matrix K_0, are three in number, each being

$$x\ ;$$

Thus the characteristic potent divisors of $[a]^{12}_{12}$ are

$$x-a_{44},\ x-a_{66},\ x^4,\ x^3,\ x,\ x,\ x.$$

Liquid motion inside certain rotating curvilinear rectangles

By

Nripendranath Sen

[*Read July 27th, 1919 and February 29th, 1920*]

1. In two important memoirs Ferrers[1] and Filon[2] have discussed the problems of the liquid motion in rotating vessels and the analogous torsion problem for boundaries determined by confocal ellipses and hyperbolas. In the present paper I have shown that it is possible to obtain a solution of this problem in its most general aspect and that the different cases discussed by these two writers are deducible as particular cases of the general problem here studied. By making use of the conjugate transformation

$$x+iy=C\,\begin{matrix}\cosh\\ \sinh\end{matrix}\,(\xi+i\eta),$$

a solution is obtained for the liquid motion in a vessel bounded by $\xi=\alpha_1,\ \alpha_2$; $\eta=\beta_1.\ \beta_2$. It is then shown that by giving particular values to $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ solutions can be obtained for a number of interesting forms of boundaries including those studied by Ferrers and Filon.

I am thankful to Dr D. N. Mallik and Dr. S. K. Banerjee for their interest in the preparation of the paper.

2. For convenience, the liquid is supposed to be contained in a cylinder of unit length and confined between two smooth parallel planes at right angles to the axis of the cylinder. The cylinder is also supposed to be rotating about the centre of the confocal system. The corresponding result for rotation about any other point can be written down very easily (see *Quart. Jour.*. Vol. 17, p. 234, 1881).

---

[1] N. M. Ferrers, "Rotating arcs of confocal ellipse and hyperbola," *Quart. Jour.*, Vol. 17, (227-44), 1881.

[2] L. N. G. Filon, "On the resistance to torsion of certain forms of shafting with special reference to the effect of key ways." *Phil. Trans.* 193A. 309-52, 1900

Let us consider the case of the liquid motion inside the rotating rectangle formed by two ellipses α_1, α_2 and two hyperbolas β_1, β_2 ; and let ω be the angular velocity of rotation.

CASE I.

3. Let $x+iy=C\cosh(\xi+i\eta)$.

Then,
$$\psi=\frac{\omega c^2}{4}\left[\cosh 2\xi+\cos 2\eta\right]. \qquad \ldots \quad (1)$$

On the boundaries

$$\begin{aligned}
&(1)\quad \xi=\alpha_1, \quad \beta_1\leqq\eta\leqq\beta_2,\\
&(2)\quad \xi=\alpha_2, \quad \beta_1\leqq\eta\leqq\beta_2.\\
&(3)\quad \eta=\beta_1, \quad \alpha_1\leqq\xi\leqq\alpha_2,\\
&(4)\quad \eta=\beta_2, \quad \alpha_1\leqq\xi\leqq\alpha_2.
\end{aligned}$$

Assume
$$\psi=\frac{\omega c^2}{4}\Bigg[\frac{\cosh 2(\xi-\gamma)\cos 2\eta}{\cosh 2\epsilon_1}+\frac{\cosh 2\xi\cos 2(\eta-\delta)}{\cos 2\epsilon_2}$$
$$+\sum_{m=0}^{m=\infty}\left\{P_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_1)\right.$$
$$\left.+\,Q_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_2)\right\}\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$
$$+\sum_{m=0}^{m=\infty}\left\{R_m\sinh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_1)\right.$$
$$\left.+\,S_m\sinh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_2)\right\}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\Bigg] \qquad (2)$$

where,
$$2\epsilon_1=\alpha_2-\alpha_1, \quad 2\epsilon_2=\beta_2-\beta_1,$$
$$2\gamma=\alpha_2+\alpha_1, \quad 2\delta=\beta_2+\beta_1. \qquad \ldots \quad (3)$$

To find P_m, Q_m, R_m, S_m.

Putting $\xi=\alpha_1$ in (1) and (2) and equating, we have,

$$\cosh 2\alpha_1+\cos 2\eta=\cos 2\eta+\frac{\cosh 2\alpha_1}{\cos 2\epsilon_2}\cos 2(\eta-\delta)$$
$$+\sum Q_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(-2\epsilon_1)\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta),$$

where $\beta_1\leqq\eta\leqq\beta_2$.

or,

$$\cosh 2\alpha_1 \left\{ -1+\frac{\cos 2(\eta-\delta)}{\cos 2\epsilon_2} \right\}$$

$$=\Sigma Q_m \sinh \frac{(2m+1)\pi\epsilon_1}{\epsilon_2} \cos \frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta),$$

where $\beta_1 \leqq \eta \leqq \beta_2$.

Multiplying both sides by $\cos \frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)d\eta$ and integrating between the limits β_1 and β_2, we have,

$$\frac{\pi}{2} Q_m \sinh \frac{(2m+1)\pi\epsilon_1}{\epsilon_2}$$

$$=(-)^m\, 2\cosh 2\alpha_1 \left[\frac{2m+1}{(2m+1)^2-\frac{16\epsilon_2^2}{\pi^2}} - \frac{1}{2m+1} \right]$$

or,

$$Q_n=\frac{(-)^m 64\,\epsilon_2^2 \cosh 2\alpha_1}{\pi\,(2m+1)\{(2m+1)^2\pi^2-16\,\epsilon_2^2\}\sinh \frac{\pi(2m+1)\epsilon_1}{\epsilon_2}} \quad \ldots \quad (4)$$

Putting $\xi=\alpha_2$ and proceeding exactly as above, we have,

$$P_n=\frac{(-)^{m+1}\, 64\,\epsilon_2^2 \cosh 2\alpha_2}{\pi\,(2m+1)\{(2m+1)^2\pi^2-16\,\epsilon_2^2\}\sinh \frac{\pi(2m+1)\epsilon_1}{\epsilon_2}} \quad \ldots \quad (5)$$

Similarly, putting $\eta=\beta_1$ in (1) and (2), equating, multiplying both sides by $\cos \frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\,d\xi$ and integrating between the limits α_1 and α_2, we have,

$$S_m=\frac{(-)^{m+1}64\epsilon_1^2\cos 2\beta_1}{\pi\,(2m+1)\{(2m+1)^2\pi^2+16\,\epsilon_1^2\}\sinh(2m+1)\frac{\pi\epsilon_2}{\epsilon_1}} \quad \ldots \quad (6)$$

Similarly, putting $\eta=\beta_2$ and proceeding exactly as above, we have,

$$R_n=\frac{(-)^m 64\epsilon_1^2\cos 2\beta_2}{\pi(2m+1)\{(2m+1)^2\pi^2+16\epsilon_1^2\}\sinh\frac{(2m+1)\pi\epsilon_2}{\epsilon_1}} \quad \ldots \quad (7)$$

Thus from (2), (3), (4), (5), (6) and (7), ψ is completly determined.

Observing that $\phi+i\psi=f(\xi+i\eta)$, we can easily deduce the expression for ϕ from that for ψ.

Thus,

$$\phi=-\frac{\omega c^2}{4}\left[\frac{\sinh 2(\xi-\gamma)\sin 2\eta}{\cosh 2\epsilon_1}+\frac{\sinh 2\xi\sin 2(\eta-\delta)}{\cos 2\epsilon_2}\right.$$

$$+\sum_{m=0}^{m=\infty}\left\{P_m\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_1)+Q_m\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_2)\right\}$$

$$\times\sin\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$

$$-\sum_{m=0}^{m=\infty}\left\{R_m\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_1)+S_m\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_2)\right\}$$

$$\left.\times\sin\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\right]\quad\ldots\quad(8)$$

The expression for the Kinetic Energy T is easily seen to be

$$\frac{32T}{\rho\omega^2c^4}=\frac{1}{2}\tan 2\epsilon_2\,(\sinh 4\alpha_2-\sinh 4\alpha_1)$$

$$-\frac{1}{2}\tanh 2\epsilon_1\,(\sin 4\beta_2-\sin 4\beta_1)$$

$$+2\epsilon_2\left\{2\tanh 2\epsilon_2+(\sinh 2\alpha_2-\sinh 2\alpha_1)\frac{\cos 2\delta}{\cos 2\epsilon_2}\right\}$$

$$-2\epsilon_1\left\{2\tan 2\epsilon_2+(\sin 2\beta_2-\sin 2\beta_1)\frac{\cosh 2\gamma}{\cosh 2\epsilon_1}\right\}$$

$$-\frac{32}{\pi}\sum_{m=0}^{m=\infty}\frac{1}{(2m+1)(M^2-4)^2}\{M(\sinh 4\alpha_2-\sinh 4\alpha_1)$$

$$-2(\cosh 4\alpha_2+\cosh 4\alpha_1+2)\coth\theta+8\cosh 2\alpha_2\cosh 2\alpha_1\,\mathrm{cosech}\,\theta\}$$

$$-\frac{32}{\pi}\sum_{m=0}^{m=\infty}\frac{1}{(2m+1)(N^2+4)^2}\{N(\sin 4\beta_2-\sin 4\beta_1)$$

$$-2(\cos 4\beta_2+\cos 4\beta_1+2)\coth\theta'+8\cos 2\beta_2\cos 2\beta_1\,\mathrm{cosech}\,\theta'\}$$

$$-\frac{64}{\pi}(\cosh 2\alpha_2+\cosh 2\alpha_1)(\cos 2\beta_2+\cos 2\beta_1)$$

$$\times\sum_{m=0}^{m=\infty}\frac{1}{2m+1}\left\{\frac{\tanh\frac{\theta}{2}}{(M^2-4)^2}+\frac{\tanh\frac{\theta'}{2}}{(N^2+4)^2}\right\},\quad\ldots\ldots(9)$$

where, $$M=\frac{(2m+1)\,\pi}{2\epsilon_2}\;;\;N=\frac{(2m+1)\,\pi}{2\epsilon_1}\;;$$

$$\theta=\frac{(2m+1)\,\pi\epsilon_1}{\epsilon_2}\;;\;\theta'=\frac{(2m+1)\,\pi\epsilon_2}{\epsilon_1}. \qquad \ldots \quad (10)$$

The following particular cases are interesting :—

(1) Liquid motion inside a rotating rectangle formed by two ellipses α_2, α_1 and the hyperbola β.

The expressions for ϕ and ψ for this case can be easily deduced by putting $-\beta_1=\beta_2=\beta$ (say) and noting that $2\epsilon_2=\beta_2-\beta_1=2\beta$ and $2\delta=\beta_2+\beta_1=0$.

(2) Liquid motion inside a rotating rectangle formed by two hyperbolas β_2, β_1 and the ellipse α.

The results for this case can be obtained by writing $\alpha_1=-\alpha_2=-\alpha$ and noting that

$$\epsilon_1=\frac{\alpha_2-\alpha_1}{2}=\alpha,\ \gamma=\frac{\alpha_2+\alpha_1}{2}=0.$$

(3) Liquid motion inside the figure formed by one ellipse α and one branch of a hyperbola β.

For this case put $\alpha_1=-\alpha_2=-\alpha$, $\beta_1=-\beta_2=-\beta$, $\epsilon_1=\alpha$, $\epsilon_2=\beta$, $\gamma=0$, $\delta=0$. This case has been studied by Filon and Ferrers. The expressions obtainted by the above substitutions are easily seen to be identical with those obtained by the latter.

(4) Liquid motion inside the curvilinear rectangle formed by two concentric circles (a_1, a_2) and two bounding radii $(\beta, -\beta)$

This can be very easily deduced by diminishing c and increasing α_1, α_2, and ξ indefinitely such that

$$c\cosh\alpha_1=c\sinh\alpha_1=a_1\;;\;c\cosh\xi=c\sinh\xi=r\;;$$

$$c\cosh\alpha_2=c\sinh\alpha_2=a_2\;;\;\xi-\gamma=\tfrac{1}{2}\log\frac{r^2}{ab}\cdot$$

$$2\epsilon_1=(\alpha_2-\alpha_1)=\log\frac{a_2}{a_1},$$

Also let $\beta_1=-\beta_2=-\beta$

so that $\epsilon_2=\beta,\quad \delta=0.$

Then after a little simplification we have,

$$\psi = \frac{\omega}{2}\left[\frac{r^2\cos 2\eta}{\cos 2\beta} + \sum_{m=0}^{m=\infty} \mathrm{M}\left\{a_2{}^2 \sinh \frac{(2m+2)\pi}{2\beta} \log\frac{r}{a_1}\right.\right.$$

$$\left.\left.-a_1{}^2 \sinh\frac{(2m+1)\pi}{2\beta}\log\frac{r}{a^2}\right\}\cos\frac{(2m+1)\pi}{2\beta}\eta\right], \quad \ldots \quad (1)$$

$$\phi = -\frac{\omega}{2}\left[\frac{r^2\sin 2\eta}{\cos 2\beta} + \sum_{m=0}^{m=\infty} \mathrm{M}\left\{a_2{}^2 \cosh \frac{(2m+1)\pi}{2\beta} \log\frac{r}{a_1}\right.\right.$$

$$\left.\left.-a_1{}^2 \cosh\frac{(2m+1)\pi}{2\beta}\log\frac{r}{a_2}\right\}\sin\frac{(2m+1)\pi}{2\beta}\eta\right], \quad \ldots \quad (2)$$

where

$$\mathrm{M} = \frac{(-)^{m+1}\ 64\beta^2}{\pi(2m+1)\left\{(2m+1)^2\pi^2 - 16\beta^2\right\}\sinh\dfrac{\pi(2m+1)\log\frac{a_2}{a_1}}{2\beta}}. \quad \ldots \quad (3)$$

The above results are simpler than those of Sir A. George Greenhill,[1] into which they can, however, be transformed after a little reduction.

(5) Liquid motion in a rotating elliptic quadrant.

This can be deduced very easily as follows. Let $a_1 = 0$, $\beta = 0$, $a_2 = a$. $\beta_2 = \frac{\pi}{2}$. Evidently, the curvilinear rectangle takes the shape of an elliptic quadrant. But the expression for ϕ and ψ and T take the undetermined forms

$$\infty - \infty$$

which, however, can be evaluated by putting

$$\beta_2 = \frac{\pi}{2} + \theta \text{ where } Lt.\ \theta = 0.$$

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[1] Sir A. G Greenhill—*Mess. Math.*, Vol. IX, p 35, 1880

The expressions for ϕ and ψ thus found out are as follows:

$$\begin{aligned}\phi = \frac{\omega c^2}{4}\Bigg[& -\frac{4}{\pi}\left\{(\xi-\alpha)\cos 2\xi \cos 2\eta - \left(\eta-\frac{\pi}{2}\right)\sinh 2\xi \sin 2\eta\right. \\ & \left. -\frac{3}{4}\sinh 2\xi \cos 2\eta - \frac{\pi}{4}\tanh \alpha \cosh 2\xi \sin 2\eta\right\} \\ & + \sum_{m=1}^{m=\infty} \mathrm{M}\left\{\cos 2(2m+1)(\xi-\alpha) - \cosh 2\alpha \cosh 2(2m+1)\xi\right\} \\ & \qquad \times \cos 2(2m+1)\eta \\ & + \sum_{m=0}^{m=\infty} \mathrm{N} \cosh \frac{(2m+1)\pi}{\alpha}\left(\eta-\frac{\pi}{4}\right) \cos \frac{(2m+1)\pi\xi}{\alpha}\Bigg], \qquad \dots (1)\end{aligned}$$

$$\begin{aligned}\psi = \frac{\omega c^2}{4}\Bigg[& -\frac{4}{\pi}\left\{(\xi-\alpha)\sinh 2\xi \sin 2\eta + \left(\eta-\frac{\pi}{2}\right)\cosh 2\xi \cos 2\eta\right. \\ & \left. -\frac{3}{4}\sin 2\eta \cosh 2\xi + \frac{\pi}{4}\tanh \alpha \sinh 2\xi \cos 2\eta\right\} \\ & + \sum_{m=1}^{m=\infty} \mathrm{M}\left\{\sinh 2(2m+1)(\xi-\alpha) - \cosh 2\alpha \sinh 2(2m+1)\xi\right\} \\ & \qquad \times \sin 2(2m+1)\eta \\ & - \sum_{m=0}^{m=\infty} \mathrm{N} \sinh \frac{(2m+1)\pi}{\alpha}\left(\eta-\frac{\pi}{4}\right) \sin \frac{(2m+1)\pi\xi}{\alpha}\Bigg], \qquad \dots (2)\end{aligned}$$

where $$\mathrm{M} = \frac{1}{\pi m(2m+1)(m+1)\sinh 2(2m+1)\alpha}, \qquad \dots (3)$$

$$\mathrm{N} = \frac{16\alpha^2}{\pi(2m+1)\{(2m+1)^2\pi^2+4\alpha^2\}\sinh \frac{(2m+1)\pi^2}{4\alpha}}. \qquad \dots (4)$$

CASE II.

Now let $x = c \cosh \xi \sin \eta$, $y = c \sinh \xi \cos \eta$.

We have then $$\psi = \frac{\omega c^2}{4}(\cosh 2\xi - \cos 2\eta), \qquad \dots (1)$$

and on the boundaries (1) $\xi=\alpha_1$, $\beta_1 \leqq \eta \leqq \beta_2$,

(2) $\xi=\alpha_2$, $\beta_1 \leqq \eta \leqq \beta_2$,

(3) $\eta=\beta_1$, $\alpha_1 \leqq \xi \leqq \alpha_2$,

(4) $\eta=\beta_2$, $\alpha_1 \leqq \xi \leqq \alpha_2$.

Assume
$$\psi=\frac{\omega c^2}{4}\left[\frac{\cosh 2\xi \cos 2(\eta-\delta)}{\cos 2\epsilon_2}-\frac{\cosh 2(\xi-\gamma)\cos 2\eta}{\cosh 2\epsilon_1}\right.$$
$$+\sum_{m=0}^{m=\infty}\left\{ P_m \sinh \frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_1)\right.$$
$$\left.+Q_m \sinh \frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_2)\right\} \times \cos \frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$
$$+\sum_{m=0}^{m=\infty}\left\{ R_m \sinh \frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_1)\right.$$
$$\left.\left.+S_m \sinh \frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_2)\right\} \times \cos \frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\right], \quad \ldots \quad (2)$$

where
$$2\epsilon_1=\alpha_2-\alpha_1, \quad 2\gamma=\alpha_2+\alpha_1, \quad \ldots \quad (3)$$
$$2\epsilon_2=\beta_2-\beta_1, \quad 2\delta=\beta_2+\beta_1.$$

Proceeding exactly as in Case I, we have,

$$P_m=\frac{(-)^{m+1}64\epsilon_2{}^2\cosh 2\alpha_2}{\pi(2m+1)\{(2m+1)^2\pi^2-16\epsilon_2{}^2\}\sinh \frac{(2m+1)\pi\epsilon_1}{\epsilon_2}},$$

$$Q_m=-\frac{P_m}{\cosh 2\alpha_2}\cosh 2\alpha_1,$$

$$R_m=\frac{(-)^{m+1}64\epsilon_1{}^2\cos 2\beta_2}{\pi(2m+1)\{(2m+1)^2\pi^2+16\epsilon_1{}^2\}\sinh \frac{(2m+1)\pi\epsilon_2}{\epsilon_1}},$$

$$S_m=-\frac{R_m}{\cos 2\beta_2}\cos 2\beta_1. \quad \ldots \quad (4)$$

Thus, ψ is completely determined from (2), (3) and (4).

Since $\phi+i\psi=f(\xi+i\eta)$, ϕ can be deduced from (2) very easily.

Thus
$$\phi=-\frac{\omega c^2}{4}\left[\frac{\sinh 2\xi \sin 2(\eta-\delta)}{\cos 2\epsilon_2}-\frac{\sinh 2(\xi-\gamma)\sin 2\eta}{\cosh 2\epsilon_1}\right.$$
$$+\sum_{m=0}^{m=\infty}\left\{P_m \cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_1)\right.$$
$$\left.+Q_m \cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_2)\right\}\sin\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$
$$-\sum_{m=0}^{m=\infty}\left\{R_m \cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_1)\right.$$
$$\left.\left.+S_m \cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_2)\right\}\sin\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\right]. \quad \ldots \quad (5)$$

The expression for the kinetic energy can be worked out as before.

The following particular cases can be deduced from the above expressions :—

(1) Liquid motion inside two ellipses and two branches of the same hyperbola β.

For this, put $2\epsilon_2=\beta$, $\delta=0$ in the above expressions for ϕ and ψ.

(2) Liquid motion inside two hyperbolas and one ellipse (α).

For this put $2\epsilon_1=2\alpha$, $\gamma=0$ in (2) and (5).

(3) Liquid motion inside one ellipse (α) two branches of the same hyperbola (β).

For this, put $\alpha_1=-\alpha_2=-\alpha$.

$\beta_1=-\beta_2=-\beta$ in (2) and (5).

So that $\epsilon_1=\alpha$, $\epsilon_2=\beta$, $\gamma=0$, $\delta=0$.

The expressions for ϕ and ψ are thus obtained. The results thus deduced are found to be identical with those of Ferrers.[1]

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[1] Ferrers.—ibid.

(*d*) Liquid motion inside a rectangular box $(2a, 2b)$ rotating about the origin with its centre at (x_1, y_1).

This is deduced very easily from the above expressions by increasing c and diminishing α, ξ, β, η, indefinitely, such that

$$\begin{aligned} &c\eta = x, \quad c\xi = y, \\ &c\epsilon_2 = a, \quad c\epsilon_1 = b, \qquad \ldots \quad (1) \\ &c\delta = x_1, \quad c\gamma = y_1, \end{aligned}$$

we thus get,

$$\begin{aligned} \phi = \omega\Big[&x_1 y - y_1 x \\ &- \sum_{m=0}^{m=\infty} P_m \sinh \pi \frac{(2m+1)(y-y_1)}{2a} \sin \frac{(2m+1)\pi}{2a}(x-x_1) \\ &+ \sum_{m=0}^{m=\infty} Q_m \sinh \frac{(2m+1)\pi}{2b}(x-x_1) \sin \frac{(2m+1)\pi}{2b}(y-y_1)\Big], \end{aligned} \qquad \ldots \quad (2)$$

$$\begin{aligned} \psi = \omega\Big[&x x_1 + y y_1 - C \\ &+ \sum_{m=0}^{m=\infty} P_m \cosh \frac{\pi(2m+1)(y-y_1)}{2a} \cos \frac{(2m+1)\pi}{2a}(x-x_1) \\ &+ \sum_{m=0}^{m=\infty} Q_m \cosh \frac{\pi(2m+1)(x-x_1)}{2b} \cos \frac{(2m+1)\pi}{2b}(y-y_1)\Big], \end{aligned} \qquad \ldots \quad (3)$$

where

$$P_m = \frac{(-)^{m+1} 16a^2}{\pi^3 (2m+1)^3 \cosh \dfrac{(2m+1)\pi b}{2a}}, \qquad \ldots \quad (4)$$

$$Q_m = \frac{(-)^{m+1} 16b^2}{\pi^3 (2m+1)^3 \cosh \dfrac{(2m+1)\pi a}{2b}},$$

C = a constant.

Putting $x_1 = 0$, $y_1 = 0$, we can at once deduce the well-known[1] result for the liquid motion inside a rectangular box rotating about its centre.

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[1] Ferrers.—*Quart. Jour.*, Vol. 15, p. 183, 1877; Vol. 17, p. 231, 1879.

The following two alternative solutions can also be easily obtained (for ψ and ϕ in Case I):—

$$\psi = \frac{\omega c^2}{4}\left[\cos 2\delta \cos 2\epsilon_2 - \sin 2\delta \sin 2\epsilon_1 \frac{\theta}{\epsilon_2} + \frac{\cosh 2\xi \cos 2\theta}{\cos 2\epsilon_2}\right.$$

$$+\frac{16}{\pi}\sin 2\epsilon_2 \sin 2\delta \sum_{m=1}^{\infty} \frac{(-)^m}{2m\left(\dfrac{m^2\pi^2}{\epsilon_2{}^2}-4\right)} \frac{\cosh \dfrac{m\pi\phi}{\epsilon_2}}{\cosh \dfrac{m\pi\epsilon_1}{\epsilon_2}} \sin \frac{m\pi}{\epsilon_2}\theta.$$

$$+\frac{16}{\pi}\sum_{m=0}^{m=\infty} \frac{(-)^{m+1}}{(2m+1)\left\{\dfrac{(2m+2)^2\,\pi^2}{4\epsilon_2{}^2}-4\right\}} \cos \frac{(2m+1)\pi\theta}{2\epsilon_2}$$

$$\times\left\{(\cosh 2\gamma \cosh 2\epsilon_1 - \cos 2\delta \cos 2\epsilon_2) \frac{\cosh \dfrac{(2m+1)\pi\phi}{2\epsilon_2}}{\cosh \dfrac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\right.$$

$$\left.\left. + \sinh 2\gamma \sinh 2\epsilon_1 \frac{\sinh \dfrac{(2m+1)\pi\phi}{2\epsilon_2}}{\sinh \dfrac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\right\}\right],$$

$$\psi = \frac{\omega c^2}{4}\left[\cosh 2\gamma \cos 2\epsilon_1 + \sinh 2\gamma \sinh 2\epsilon_1 \frac{\phi}{\epsilon_1} + \frac{\cosh 2\phi \cos 2(\theta+\delta)}{\cosh 2\epsilon_1}\right.$$

$$+\frac{16}{\pi}\sinh 2\gamma \sinh 2\epsilon_1 \sum_{m=1}^{m=\infty} \frac{(-)^m}{2m\left\{\dfrac{m^2\pi^2}{\epsilon_1{}^2}+4\right\}} \frac{\cosh \dfrac{m\pi\theta}{\epsilon_1}}{\cosh \dfrac{m\pi\epsilon_2}{\epsilon_1}} \sin \frac{m\pi\phi}{\epsilon_1}$$

$$+\frac{16}{\pi}\sum_{m=0}^{m=\infty} \frac{(-)^{m+1}}{(2m+1)\left\{\dfrac{(2m+1)^2\pi^2}{4\epsilon_1{}^2}+4\right\}} \cos \frac{(2m+1)\pi\phi}{2\epsilon_1}$$

$$\times\left\{(\cosh 2\gamma \cosh 2\epsilon_1 - \cos 2\delta \cos 2\epsilon_2) \frac{\cosh \dfrac{(2m+1)\pi\theta}{2\epsilon_1}}{\cosh \dfrac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\right.$$

$$\left.\left. + \sin 2\delta \sin 2\epsilon_2 \frac{\sinh \dfrac{(2m+1)\pi\theta}{2\epsilon_1}}{\sinh \dfrac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\right\}\right],$$

where $\phi = \xi - \gamma;\ \theta = \eta - \delta.$

The corresponding expressions for ϕ can be easily written down.

Similar alternative expressions for ϕ and ψ can also be obtained for Case II and from these Filon's solutions may be deduced by putting

$$\gamma=0,\ \epsilon_1=\alpha.$$

The proof of these expressions for ϕ and ψ depends on the identity

$$\frac{\pi}{16}\,\frac{\cosh 2\phi \sin 2\theta}{\cosh 2\epsilon_1 \sin 2\epsilon_2}-\frac{\pi\theta}{16\epsilon_2}+\sum_{m=0}^{m=\infty}\frac{(-)^m}{2m+1},$$

$$\frac{\sinh\dfrac{(2m+1)\pi\theta}{2\epsilon_1}}{\left\{\dfrac{(2m+1)^2\pi^2}{4\epsilon_1{}^2}+4\right\}\sinh\dfrac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}\phi$$

$$+\sum_{m=1}^{m=\infty}\frac{(-)^m}{\left(2m\dfrac{m^2\pi^2}{\epsilon_2{}^2}-4\right)}\,\frac{\cosh\dfrac{m\pi\phi}{\epsilon_2}}{\cosh\dfrac{m\pi\epsilon_1}{\epsilon_2}}\sin\frac{m\pi\theta}{\epsilon_2}=0,$$

and two others.

This identity may be proved by starting from the expression

$$u=-\frac{\pi\theta}{4\epsilon_2}+\sum_{m=0}^{m=\infty}\frac{(-)^m}{2m+1}\,\frac{\sinh\dfrac{(2m+1)\pi\theta}{2\epsilon_1}}{\sinh\dfrac{(2m+1)\pi\epsilon_2}{\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}\phi$$

$$+\sum_{m=1}^{m=\infty}\frac{(-)^{m-1}}{2m}\,\frac{\cosh\dfrac{m\pi}{\epsilon_2}\phi}{\cosh\dfrac{m\pi\epsilon_1}{\epsilon_2}}\sin\frac{m\pi\theta}{\epsilon_2},$$

and noting that the potential function u vanishes on the boundaries defined by ($\xi=\alpha_1, \alpha_2$; $\eta=\beta_1, \beta_2$) and therefore at every point inside the region enclosed by these boundaries. Now since the two operators $\dfrac{\partial^2}{\partial\theta^2}+4$ and $-\dfrac{\partial^2}{\partial\phi^2}+4$ are identical, if we perform the operations $\left(\dfrac{\partial^2}{\partial\theta^2}+4\right)^{-1}$ and $\left(-\dfrac{\partial^2}{\partial\phi^2}+4\right)^{-1}$ on both sides of the above expression with

u put equal to zero, we easily obtain

$$0=-\frac{\pi\theta}{16\epsilon_2}+\sum_{m=0}^{m=\infty}\frac{(-)^m}{(2m+1)}\frac{\sinh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\left\{\frac{(2m+1)^2\pi^2}{4\epsilon_1^{\,2}}+4\right\}\sinh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}\phi$$

$$+\sum_{m=1}^{m=\infty}\frac{(-)^m}{2m}\cdot\frac{\cosh\frac{m\pi\phi}{\epsilon_2}}{\left\{\frac{m^2\pi^2}{\epsilon_2^{\,2}}-4\right\}\cosh\frac{m\pi\epsilon_1}{\epsilon_2}}\sin\frac{m\pi\theta}{\epsilon_2}+f(\phi)\sin 2\theta$$

$$=-\frac{\pi\theta}{16\epsilon_2}+\sum_{m=0}^{m=\infty}\frac{(-)^m}{(2m+1)}\frac{\sinh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\left\{\frac{(2m+1)^2\pi^2}{4\epsilon_1^{\,2}}+4\right\}\sinh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi\phi}{2\epsilon_1}$$

$$+\sum_{m=0}^{m=\infty}\frac{(-)^m}{2m}\cdot\frac{\cosh\frac{m\pi\phi}{\epsilon_2}}{\left(\frac{m^2\pi^2}{\epsilon_2^{\,2}}-4\right)\cosh\frac{m\pi\epsilon_1}{\epsilon_2}}\sin\frac{m\pi\theta}{\epsilon_2}+\mathrm{F}(\theta)\cosh 2\phi,$$

where $-\epsilon_1<\phi<\epsilon_1,\ -\epsilon_2<\theta<\epsilon_2.$

Hence $f(\phi)=c\cosh 2\phi$, $\mathrm{F}(\theta)=c\sin 2\theta$ and by putting $\phi=\epsilon_1$, $\theta=\epsilon_2$, we obtain $c=\dfrac{\pi}{16\cosh 2\epsilon_1\sin 2\epsilon_2}$ which proves the identity. A similar identity is obtained by starting with an expression for u in which θ, ϕ and ϵ_1, ϵ_2 have been interchanged. If on the other hand we start with the expression,

$$u=\sum_{m=0}^{m=\infty}\frac{(-)^m}{2m+1}\left\{\frac{\cosh\frac{(m+1)\pi\phi}{2\epsilon_2}}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\cos\frac{(2m+1)\pi\theta}{2\epsilon_2}\right.$$

$$\left.+\frac{\cosh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}\phi\right\},$$

we get,

$$\frac{\pi}{16}\left[1-\frac{\cosh 2\phi \cos 2\theta}{\cosh 2\epsilon_1 \cos 2\epsilon_2}\right] = \sum_{m=0}^{m=\infty}\left\{\frac{(-)^{m+1}}{(2m+1)\left(\frac{(2m+1)^2\pi^2}{4\epsilon_2^{\,2}}-4\right)} \frac{\cosh\frac{(2m+1)\pi\phi}{2\epsilon_2}}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}} \cos\frac{(2m+1)\pi\theta}{2\epsilon_2} + \frac{(-)^m}{(2m+1)\left\{\frac{(2m+1)^2\pi^2}{4\epsilon_1^{\,2}}+4\right\}} + \frac{\cosh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}} \cos\frac{(2m+1)\pi\phi}{2\epsilon_1}\right\},$$

(*vide* Ferrers, *l.c.*).

On the Horpolhode[1]

By

Satyendranath Basu

[*Read August 31st, 1919*]

M. de Sparre has shown that the horpolhode of Poinsot contains no point of inflexion ; his proof was based upon the properties of elliptic functions. Various other proofs of the theorem have since been given by Mannheim, Saint-Germain,[2] Routh,[3] Le-Cornu[4] and others.

The following simple proof of the theorem is based solely on the dynamical considerations of the Poinsot motion; hence it might appear interesting and instructive.

Let A, B, C be the three principal moments of inertia of a body fixed at a point O; let OI and OI′ be the two instantaneous axes, at time t and $t+dt$, I, I′, etc., on the invariable plane trace out the horpolhode, length of OI, OI′, etc., being proportional to the resulting angular velocity at all times.

At a point of inflexion of the horpolhode, there will be a stationary tangent, *i.e.*, the total change of the vector II′ will be then along its own length.

Remarking that II′ has components proportional to $\dot{\omega}_1 dt$, $\dot{\omega}_2 dt$, $\dot{\omega}_3 dt$, along the three moving principal axes, the condition reduces to

$$(1)\quad \frac{\ddot{\omega}_1-\dot{\omega}_2\omega_3+\dot{\omega}_3\omega_2}{\dot{\omega}_1}=\frac{\ddot{\omega}_2-\dot{\omega}_3\omega_1+\dot{\omega}_1\omega_3}{\dot{\omega}_2}=\frac{\ddot{\omega}_3-\dot{\omega}_1\omega_2+\dot{\omega}_2\omega_1}{\dot{\omega}_3}$$

$$=\lambda \text{ say.}$$

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[1] An exactly similar proof of the theorem discussed in this paper has been given by Prof. W. Van der Woude in a paper entitled "Over de Herpolöide Van Poinsot," published in *Nieuw Archief voor Wiskunde*, Tweede Reeks, Deel XIII, Eerste Stuk, pp. 94, 95 (1919). It appears however from an enquiry that this journal was not received in the library of the Society before the beginning of November, 1919.—S.K.B.

[2] Compte Rendu—1885.

[3] Advanced Dynamics.

[4] Bulletin de Soc. Math. de France—1906.

We have from Euler's equations

$$A\dot{\omega}_1=(B-C)\omega_2\omega_3,$$

$$B\dot{\omega}_2=(C-A)\omega_3\omega_1,$$

$$C\dot{\omega}_3=(A-B)\omega_1\omega_2\ ;$$

from which

$$A\ddot{\omega}_1=(B-C)(\dot{\omega}_2\omega_3+\dot{\omega}_3\omega_2),$$

$$B\ddot{\omega}_2=(C-A)(\dot{\omega}_3\omega_1+\omega_1\dot{\omega}_3),$$

$$C\ddot{\omega}_3=(A-B)(\dot{\omega}_1\omega_2+\dot{\omega}_2\omega_1).$$

Substituting in (1) we have

$$\frac{\dot{\omega}_3}{\omega_3}(A+B-C)-\frac{\dot{\omega}_2}{\omega_2}(C+A-B)=\lambda(B-C)$$

and two similar equations.

So that

$$\frac{\dot{\omega}_1}{\omega_1}(B-C)(B+C-A)+\frac{\dot{\omega}_2}{\omega_2}(C-A)(C+A-B)+\frac{\dot{\omega}_3}{\omega_3}(A-B)(A+B-C)=0,$$

But remembering that

$$A\omega_1\dot{\omega}_1+B\omega_2\dot{\omega}_2+C\omega_3\dot{\omega}_3=0,$$

$$A^2\omega_1\dot{\omega}_1+B^2\omega_2\dot{\omega}_2+C^2\omega_3\dot{\omega}_3=0,$$

we have

$$\frac{\omega_1\dot{\omega}_1}{BC(B-C)}=\frac{\omega_2\dot{\omega}_2}{CA(C-A)}=\frac{\omega_3\dot{\omega}_3}{AB(A-B)}$$

hence substituting, we get as a necessary condition for the point of inflexion.

$$\frac{BC(B-C)^2(B+C-A)}{\omega_1^{\ 2}}+\frac{CA(C-A)^2(C+A-B)}{\omega_2^{\ 2}}$$

$$+\frac{AB(A-B)^2(A+B-C)}{\omega_3^{\ 2}}=0$$

which is obviously impossible because $B+C-A$, $C+A-B$, $A+B-C$, are all positive, so that the left-hand side is an essentially positive quantity.

On a Generalization of Neumann's Expansion in a Series of Bessel Functions

BY

ABANIBHUSHAN DATTA

[*Read February 29th, 1920*]

1. Neumann has shown that any arbitrary function, regular at the origin, can be expanded in a series of Bessel Functions of the form:

$$f(z)=a_0 J_0(z)+a_1 J_1(z)+\ldots,$$

where the co-efficients a_0, a_1, a_2, ... are independent of z. In the present paper Neumann's method has been usefully employed to obtain the expansion of any arbitrary function in terms of Bessel Function in the following more general forms:

$$f(z)=\Sigma b_n \; J_{n+a}(kz), \qquad \ldots \quad \text{(I)}$$

$$f(z)=\Sigma b_n \frac{J_{n+a}(kz)}{z^a}, \qquad \ldots \quad \text{(II)}$$

where the co-efficients b_n, etc., are independent of z, and a is any-positive integer or fraction as the case may be. An interesting application of the results obtained here consists in obtaining defferent forms of expansion for the same function in terms of Bessel Functions of integral or fractional orders. Some simple examples have been given towards the end of this paper. It may be remarked that a series in Bessel Functions of increasing orders is usually convergent. It may however be necessary to test the convergence in some of the particular cases given hereafter.

2. Let us first consider the expansion in the form (I).

Let
$$f(z)=\sum_{n=o}^{n=\infty} b_n J_{n+a}(kz),$$
where a is an integer.

Let us try to determine the co-efficients by expanding both sides of the equation as power series in z and equating co-efficients of the several powers of z.

Now $$f(z)=f(o)+zf'(o)+\frac{z^2}{2!}f''(o)+\dots$$

and $$J_{n+a}(kz)=\left(\frac{k}{2}\right)^{n+a}\frac{z^{n+a}}{\Gamma(n+a+1)}\left[1-\left(\frac{k}{2}\right)^2\frac{z^2}{(n+a+1)}\right.$$
$$\left.+\left(\frac{k}{2}\right)^4\frac{z^4}{2\,!\,(n+a+1)(n+a+2)}-\dots\right].$$

Hence $f(o)=o$, $f'(o)=o\dots f^r(o)=o$ where $r<a$.

When $r=a$,

$$\frac{f^a(o)}{a\,!}=\left(\frac{k}{2}\right)^a\frac{1}{\Gamma(a+1)}\,b_0$$

$$\frac{f^{a+1}(o)}{(a+1)!}=\left(\frac{k}{2}\right)^{1+a}\frac{1}{\Gamma(a+2)}\,b_1$$

$$\frac{f^{a+2}(o)}{(a+2)!}=-\left(\frac{k}{2}\right)^{a+2}\frac{b_0}{\Gamma(a+2)}+\left(\frac{k}{2}\right)^{a+2}\frac{b_2}{\Gamma(a+3)}$$

$$\frac{f^{a+3}(o)}{(a+3)!}=-\left(\frac{k}{2}\right)^{a+3}\frac{b_1}{\Gamma(a+3)}+\left(\frac{k}{2}\right)^{a+3}\frac{b_3}{\Gamma(a+4)}$$

$$\frac{f^{a+4}(o)}{(a+4)!}=\left(\frac{k}{2}\right)^{a+4}\frac{b_0}{2\,!\,\Gamma(a+3)}-\left(\frac{k}{2}\right)^{a+4}\frac{b_2}{\Gamma(a+4)}$$
$$+\left(\frac{k}{2}\right)^{a+4}\frac{b_4}{\Gamma(a+5)}$$

... ...

$$\frac{f^{a+2n}(o)}{(a+2n)!}=(-1)^n\left(\frac{k}{2}\right)^{a+2n}\times\left[\frac{b_0}{n\,!\,\Gamma(a+n+1)}\right.$$
$$\left.-\frac{b_2}{(n-1(!\Gamma(n+a+2)}+\dots+(-1)^n\frac{b_{2n}}{\Gamma(2n+a+1)}\right]$$

$$\frac{f^{a+2n+1}(o)}{(a+2n+1)}=(-1)^n\left(\frac{k}{2}\right)^{2n+a+1}\left[\frac{b_1}{n!\Gamma(n+a+2)}\right.$$
$$\left.-\frac{b_3}{(n-1)\,!\,\Gamma(n+a+3)}+\dots+(-1)^n\frac{b_{2n+1}}{\Gamma(2n+a+2)}\right].$$

From the above identities, we have

$$b_0=\left(\frac{2}{k}\right)^a f^a(o)$$

$$b_1=\left(\frac{2}{k}\right)^{1+a} f^{a+1}(o)$$

$$b_2=\left(\frac{2}{k}\right)^{a+2} \Gamma(a+3)\left[\left(\frac{k}{2}\right)^{a+2} \frac{b_0}{\Gamma(a+2)}+\frac{f^{a+2}(o)}{(n+2)\,!}\right]$$

$$=\left(\frac{2}{k}\right)^a\left[(a+2\, f^a(o)+\left(\frac{2}{k}\right)^2 f^{a+2}(o)\right]$$

$$b_3=\left(\frac{2}{k}\right)^{a+4}\Gamma(a+4)\left[\left(\frac{k}{2}\right)^{a+3} \frac{b_1}{\Gamma(a+3)}+\frac{f^{a+3}(o)}{(a+3)\,!}\right]$$

$$=\left(\frac{2}{k}\right)^{a+1}\left[(a+3\, f^{a+1}(o)+\left(\frac{2}{k}\right)^2 f^{a+3}(o)\right]$$

$$b_4=(-1)^3\Gamma(a+5)\left[\frac{b_0}{2\,!\,\Gamma(a+3)}-\frac{b_2}{\Gamma(a+4)}\right.$$

$$\left.+(-)\left(\frac{2}{k}\right)^{a+4} \frac{f^{a+4}(o)}{(a+4)\,!}\right]$$

$$=(-)^3\left(\frac{2}{k}\right)^a\left[f^a(o)\left\{\frac{\Gamma(a+5)}{2\,!\,\Gamma(a+3)}-\frac{\Gamma(a+5)}{\Gamma(a+4)}(a+2)\right\}\right.$$

$$\left.-(a+4)f^{a+2}(o)\left(\frac{2}{k}\right)^2+(-)\left(\frac{2}{k}\right)^4 f^{a+4}(o)\right]$$

... ...

$$b_{2n}=(-)^{n+1}\Gamma(2n+a+1)\left[\frac{b_0}{n\,!\,\Gamma(n+a+1)}-\frac{b_2}{(n-1)\,!\,\Gamma(n+a+2)}\right.$$

$$\left.+\ \dots\ +(-1)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{f^{2n+a}(o)}{(2n+a)\,!}\right]$$

$$\doteq(-1)^{n+1}\left(\frac{2}{k}\right)^a\left[\Gamma(2n+a+1)f^a(o)\left\{\frac{1}{n\,!\,\Gamma(n+a+1)}\right.\right.$$

$$-\frac{\Gamma(a+3)}{(n-1)\,\Gamma(n+a+2)}-\left(\frac{\Gamma(a+5)}{2\,!\,\Gamma(a+3)}\right.$$

$$\left.\left.-\frac{\Gamma(a+5)}{\Gamma(a+4)}\cdot\frac{\Gamma(a+3)}{\Gamma(a+2)}\right)\frac{1}{(n-2)\,!\,\Gamma(n+a+3)}-\ \cdots\right\}$$

$$-\left(\frac{2}{k}\right)^{2}\Gamma(2n+a+1)f^{a+2}(o)\left\{\frac{1}{(n-1)\,!\,\Gamma(a+n+2)}\right.$$

$$\left.-\frac{\Gamma(a+5)}{\Gamma(a+4)}\cdot\frac{1}{(n-2)\,!\,\Gamma(n+a+3)}-\ \dots\right\}$$

$$+\qquad\dots\qquad\dots\qquad\dots$$

$$+(-)^{n-2}\left(\frac{2}{k}\right)^{2n-4-a}f^{2n+a-4}(o)\left\{\frac{\Gamma(2n+a+1)}{\Gamma(2n+a-1)2\,!}\right.$$

$$\left.-\frac{\Gamma(2n+a+1)}{\Gamma(2n+a)}\cdot\frac{\Gamma(2n+a-1)}{\Gamma(2n+a-2)}\right\}$$

$$+(-)^{n-1}\left(\frac{2}{k}\right)^{2n-2}f^{2n+a-2}(o)\ \frac{\Gamma(2n+a+1)}{\Gamma(2n+a)}$$

$$\left.+(-)^{n-1}\left(\frac{2}{k}\right)^{2n}f^{2n+a}(o)\ \right]$$

$$b_{2n+1}=(-)^{n-1}\Gamma(2n+a+2)\left[\frac{b_1}{n\,!\,\Gamma(n+a+2)}-\frac{b_3}{(n-1)\,!\,\Gamma(n+a+3)}\right.$$

$$\left.+\ \dots\ +(-)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{f^{2n+a-1}(o)}{\Gamma(2n+a+1)}\right]$$

$$=(-)^{n+1}\left(\frac{2}{k}\right)^{1+a}\left[\Gamma(2n+a+2)f^{a+1}(o)\left\{\frac{1}{n\,!\,\Gamma(n+a+2)}\right.\right.$$

$$-\frac{\Gamma(a+4)}{(n-1)\,!\,\Gamma(n+a+3)\Gamma(a+3)}-\left(\frac{\Gamma(a+6)}{2\,!\,\Gamma(a+4)}\right)$$

$$\left.-\frac{\Gamma(a+b)}{\Gamma(a+5)}\cdot\frac{\Gamma(a+4)}{\Gamma(a+3)}\right)\frac{1}{(n-2)\,!\,\Gamma(n+a+4)}-\ \dots\Bigg\}$$

$$+\left(\frac{2}{k}\right)^{2}\Gamma(2n+a+2)f^{a+3}(o)\left\{\frac{1}{(n-1)\,!\,\Gamma(n+a+3)}\right.$$

$$\left.-\frac{\Gamma(a+b)}{\Gamma(a+5)(n-2)\,!\,\Gamma(n+a+4)}-\ \dots\right\}$$

$$\dots\qquad\dots\qquad\dots\qquad\dots\qquad\dots$$

$$+(-)^{n-2}\left(\frac{2}{k}\right)^{2r-4}f^{2n+a-3}(o)\left\{\frac{(2n+a+1)(2n+a)}{2\,!}\right.$$

$$\left.-(2n+a+1)(2n+a-1)\right\}+(-)^{n-1}\left(\frac{2}{k}\right)^{2n-2}f^{2n+a-2}(o).(2n+a+1)$$

$$\left.+(-1)^{n-1}f^{2n+a+1}(o)\left(\frac{2}{k}\right)^{2n}\right].$$

3. Next, let us take up the expansion in the form (II)

$$f(z)=\sum_{n=o}^{n=\infty} b_n \frac{J_{n+a}(kz)}{z^a}$$

where a is any positive integer or fraction.

Here we shall follow a method similar to that in §.2.

As, $$z^{-a}J_{n+a}(kz)=\left(\frac{k}{2}\right)^{n+a}\frac{z^n}{\Gamma(n+a+1)}\left[1-\left(\frac{k}{2}\right)^2\frac{z^2}{n+a+1}\right.$$
$$\left.+\left(\frac{k}{2}\right)^4\frac{z^4}{(n+a+1)(n+a+2)2\,!}-\cdots\right]$$

we get,

$$f(o)=\left(\frac{k}{2}\right)^a\frac{b_0}{\Gamma(a+1)}$$

$$f'(o)=\left(\frac{k}{2}\right)^{1+a}\frac{b_1}{\Gamma(a+2)}$$

$$\frac{f''(o)}{2\,!}=-\left(\frac{k}{2}\right)^{2+a}\frac{b_0}{\Gamma(a+2)}+\left(\frac{k}{2}\right)^{2+a}\frac{b_2}{\Gamma(a+3)}$$

$$\frac{f'''(o)}{3\,!}=-\left(\frac{k}{2}\right)^{3+a}\frac{b_1}{\Gamma(a+3)}+\left(\frac{k}{2}\right)^{3+a}\frac{b_3}{\Gamma(a+4)}$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$\frac{f^{2n}(o)}{2n\,!}=(-)^n\left(\frac{k}{2}\right)^{2n+a}\left[\frac{b_0}{n\,!\,\Gamma(n+a+1)}-\frac{b_2}{(n-1)\,!\,\Gamma(n+a+2)}\right.$$
$$\left.+\ \ldots\ +(-)^n\frac{b_{2n}}{\Gamma(2n+a+1)}\right]$$

$$\frac{f^{2n+1}(o)}{(2n+1)\,!}=(-)^n\left(\frac{k}{2}\right)^{2n+a+1}\left[\frac{b_1}{n\,!\,\Gamma(n+a+2)}\right.$$
$$\left.-\frac{b_3}{(n-1)\,!\,\Gamma(n+a+3)}+(-)^n\frac{b_{2n+1}}{\Gamma(2n+a+2)}\right].$$

From the above identities, we have

$$b_0=\left(\frac{2}{k}\right)^a\Gamma(a+1)f(o)$$

$$b_1=\left(\frac{2}{k}\right)^{1+a}\Gamma(a+2)\,.\,f'(o)$$

$$b_2=\left(\frac{2}{k}\right)^{2+a}\Gamma(a+3)\left[\left(\frac{k}{2}\right)^{2+a}\frac{b_0}{\Gamma(a+2)}+\frac{f''(o)}{2\,!}\right]$$

$$=\left(\frac{2}{k}\right)^{a}\Gamma(a+3)\left[\frac{\Gamma(a+1)}{\Gamma(a+2)}f(o)+\left(\frac{2}{k}\right)^{2}\frac{f''(o)}{2\,!}\right]$$

$$b_3=\left(\frac{2}{k}\right)^{3+a}\Gamma(a+4)\left[\left(\frac{k}{2}\right)^{3+a}\frac{b_1}{\Gamma(a+3)}+\frac{f'''(o)}{3\,!}\right]$$

$$=\left(\frac{2}{k}\right)^{a+1}\Gamma(a+4)\left[\frac{\Gamma(a+2)}{\Gamma(a+3)}f'(o)+\left(\frac{2}{k}\right)^{2}\frac{f'''(o)}{3\,!}\right]$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$b_{2n}=(-)^{n+1}\Gamma(2n+a+1)\left[\frac{b_0}{n\,!\,\Gamma(n+a+1)}-\frac{b_2}{(n-1)\,!\,\Gamma(n+a+2)}\right.$$

$$\left.+\;\ldots\;+(-)^{n-1}\frac{f^{2n}(o)}{2n\,!}\right]$$

$$=(-)^{n+1}\left(\frac{2}{k}\right)^{a}\Gamma(2n+a+1)\left[f(o)\left\{\frac{\Gamma(a+1)}{n\,!\,\Gamma(n+a+1)}\right.\right.$$

$$\left.-\frac{\Gamma(a+3)\Gamma(a+1)}{(n-1)\,!\,\Gamma(a+2)\Gamma(n+a+2)}+\;\ldots\right\}$$

$$-\left(\frac{2}{k}\right)^{2}f''(o)\left\{\frac{\Gamma(a+3)}{(n-1)\,!\,2\,!\,\Gamma(n+a+2)}-\;\ldots\right\}$$

$$\left.+\;\ldots\;+(-)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{f^{2n}(o)}{2n\,!}\right]$$

$$b_{2n+1}=(-1)^{n+1}\left(\frac{2}{k}\right)^{1+a}\Gamma(2n+a+2)\left[f(o)\left\{\frac{\Gamma(a+2)}{n\,!\,\Gamma(n+a+2)}\right.\right.$$

$$\left.-\frac{\Gamma(a+4)\Gamma(a+2)}{(n-1)\,!\,\Gamma(a+3)\Gamma(a+n+3)}+\;\ldots\right\}$$

$$-\left(\frac{2}{k}\right)^{2}f'''(o)\left\{\frac{\Gamma(a+4)}{3\,!\,(n-1)\,!\,\Gamma(n+a+3)}-\;\ldots\right\}$$

$$\left.+\;\ldots\;+(-)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{f^{2n+1}(o)}{(2n+1)!}\right].$$

These co-efficients can be made to take simpler forms if we introduce the functions $O_1(z)$, $O_2(z)$, $O_3(z)$, etc., defined by the following formulae :—

$$O_{2n}(z)=(-)^{n+1}\left(\frac{2}{k}\right)^{a}\Gamma(2n+a+1)\left[\frac{1}{z}\left\{\frac{\Gamma(a+1)}{n\,!\,\Gamma(n+a+1)}\right.\right.$$

$$\left.-\ \cdots\right\}-\left(\frac{2}{k}\right)^{2}\frac{1}{z^{3}}\left\{\frac{\Gamma(a+3)}{(n-1)\,!\,2\,!\,\Gamma(n+a+2)}-\ \cdots\right\}$$

$$\left.+\ \cdots+(-)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{2n\,!}{2n\,!\,z^{2n+1}}\right]$$

$$O_{2n+1}(z)=(-)^{n+1}\left(\frac{2}{k}\right)^{1+a}\Gamma(2n+a+2)\left[\frac{1}{z^{2}}\left\{\frac{\Gamma(a+2)}{n\,!\,\Gamma(n+a+2)}\right.\right.$$

$$\left.-\ \cdots\right\}-\frac{1}{z^{4}}\left(\frac{2}{k}\right)^{2}\left\{\frac{\Gamma(a+4)}{3\,!\,(n-1)\,!\,\Gamma(n+a+2)}-\ \cdots\right\}$$

$$\left.+(-1)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{(2n+1)\,!}{(2n+1)\,!\,z^{2n+2}}\right]$$

for then it is easily seen that b_{2n} is equal to the residue of the function $O_n(t)f(t)$ at the point $t=o$.

We have thus the general expansion

$$f(z)=\sum_{n=o}^{\infty} b_n J_{n+a}(kz)\ .\ z^{-a}$$

where $b_0=\left(\frac{2}{k}\right)^{a}\Gamma(a+1)f(o)$ and $b_n(n>o)$ is the residue of the function $O_n(t)f(t)$ at the point $t=o$, so that $b_n=\frac{1}{2\pi i}\int_a O_n(t)f(t)dt$ where a is any simple contour surrounding the origin.

4. Some interesting examples can be given in illustration of the above expansions if we confine our attention to simple functions. A few of these are given below.

(*i*) Let us take as an example the expansion of cos z. Let $k=1$.

Form I.

Here $f(o)=1.\ f'(o)=o,\ f''(o)=-1.\ f'''(o)=o$, etc.

$$b_0=1,\ b_2=2\,!\left[-\frac{2^2}{2!}+\frac{1}{1!}\right]=-2,\ b_4=4\,!\left[\frac{2}{4!}-\frac{1}{2\,!\,2\,!}-\frac{2}{3\,!}\right]$$

$$=2, \text{ etc.}$$

Hence $\cos z = J_0(z) - 2J_2(z) + 2J_4(z) - \ldots$

Form II.

Case (1). Let $a = \frac{1}{2}$

Then $b_0 = 2^{\frac{1}{2}}\,\Gamma(\frac{3}{2}) = \frac{\sqrt{2\pi}}{2}$, $b_1 = o$,

$$b_2 = \sqrt{2}\;\Gamma(3+\tfrac{1}{2})\left[\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})} - 2^2\cdot\tfrac{1}{2}\right] = -\tfrac{5}{2}\sqrt{2\pi},$$

$$b_3 = o,$$

$$b_4 = -\sqrt{2}\;\Gamma(5+\tfrac{1}{2})\left[\left\{\frac{\Gamma(\frac{5}{2})}{2\,!\,\Gamma(3+\frac{1}{2})} - \frac{\Gamma(3+\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2+\frac{1}{2})\Gamma(4+\frac{1}{2})}\right\} + 2^2\frac{\Gamma(3+\frac{1}{2})}{\Gamma(4+\frac{1}{2})2\,!} - 2^4\cdot\frac{1}{4\,!}\right]$$

$$= \tfrac{9}{2}\sqrt{2\pi}.$$

Hence $$z^{\frac{1}{2}}\cos z = \sqrt{2\pi}\left[\tfrac{1}{2}J_{\frac{1}{2}}(z) - \tfrac{5}{2}J_{\frac{5}{2}}(z) + \tfrac{9}{2}J_{\frac{9}{2}}(z) - \ldots\right].$$

Case (2), Let $a = \frac{1}{3}$.

Here $b_0 = 2^{\frac{1}{3}}\Gamma(\frac{4}{3}) = 2^{\frac{1}{3}}\cdot\frac{1}{3}\Gamma(\frac{1}{3})$

$$b_2 = 2^{\frac{1}{3}}\;\Gamma(3+\tfrac{1}{3})\left[\frac{\Gamma(1+\frac{1}{3})}{\Gamma(2+\frac{1}{3})} - 2^2\cdot\frac{1}{2\,!}\right] = -2^{\frac{1}{3}}\;\tfrac{35}{27}\Gamma(\tfrac{1}{3})$$

$$b_4 = -2^{\frac{1}{3}}\;\Gamma(5+\tfrac{1}{3})\left[\left\{\frac{\Gamma(1+\frac{1}{3})}{2\,!\,\Gamma(3+\frac{1}{3})} - \frac{\Gamma(3+\frac{1}{3})\Gamma(1+\frac{1}{3})}{\Gamma(2+\frac{1}{3})\Gamma(4+\frac{1}{3})}\right\} + 2^2\cdot\frac{\Gamma(3+\frac{1}{3})}{\Gamma(4+\frac{1}{3})2\,!} - 2^4\cdot\frac{1}{4\,!}\right] = 2^{\frac{1}{3}}\;\frac{1430}{729}\;\Gamma(\tfrac{1}{3}).$$

Hence $$z^{\frac{1}{3}}\cos z = 2^{\frac{1}{3}}\;\Gamma(\tfrac{1}{3})\left[\tfrac{1}{3}J_{\frac{1}{3}}(z) - \frac{35}{27}J_{\frac{\text{[illegible]}}{3}}(z) + \frac{1430}{729}J_{\frac{13}{3}}(z) - \ldots\right]$$

(*ii*) Now let us take $f(z) = z^n$. Let $k = 1$

Here $f(o) = o$, $f'(o) = o$, $f''(o) = o \ldots f^{n-1}(o) = o$,

$$f^n(o) = n\,!,\; f^{n+r}(o) = o\;(r > o).$$

$$b_0=2^n\ .\ n\,!$$

$$b_1=o$$

$$b_2=2^n(n+2)\ .\ n\,!$$

$$b_3=o$$

$$b_4=-2^n\ .\ n\,!\left\{\frac{(n+3)\,(n+4)}{2\,!}-(n+4).(n+2)\right\}$$

$$=2^n\ .\ n\,!\ \frac{(n+4)\,(n+1)}{2\,!}.$$

$$b_6=2^n\ .\ n\,!\left\{\frac{(n+6)\,!}{3\,!\,(n+3)\,!}-\frac{(n+2)(n+6)\,!}{2\,!\,(n+4)\,!}\right.$$

$$\left.+\frac{(n+4)(n+1)(n+6)\,!}{2\,!\,(n+5)\,!}\right\}$$

$$=2^n\ .\ n\,!\ \frac{(n+6)(n+2)(n+1)}{3\,!}.$$

Hence $z^n=2^n\ .\ n\,!\left\{J_n(z)+(n+2)J_{n+2}(z)+\frac{(n+4)(n+1)}{2\,!}J_{n+4}(z)\right.$

$$\left.+\frac{(n+6)(n+2)(n+1)}{3\,!}J_{n+6}(z)+\ \dots\right\}.$$

(*iii*) Let us now consider the function e^z.

Form I. Let $k=1$

Here $b_0=2^0\ .\ f(o)=1$

$$b_1=2\ .\ f(o)=2$$

$$b_2=[2\ .\ 1+2^2\ .\ 1]=6$$

$$b_3=2\cdot[3\cdot1+2\cdot1]=14$$

$$b_4=-\left[\frac{3\ .\ 4}{2}-4\ .\ 2-4\ .\ 2^2-2^4\right]=34.$$

Hence $e^z=J_0(z)+2J_1(z)+6J_2(z)+14J_3(z)+34J_4(z)+\dots,$

Form II. Let $a=\frac{1}{2}$, $k=1$

$$b_0=2^{\frac{1}{2}}\ \Gamma(\tfrac{3}{2})=\frac{\sqrt{2\pi}}{2}.$$

$$b_1=2^{\frac{3}{2}}\ \Gamma(\tfrac{5}{2})=\sqrt{2\pi}\cdot\tfrac{3}{2}.$$

$$b_2=2^{\frac{1}{2}}\ \Gamma(\tfrac{7}{2})\left[\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})}+2^2\ \frac{1}{2!}\right]=\sqrt{2\pi}.5$$

$$b_3=2^{\frac{7}{2}}\ \Gamma(\tfrac{9}{2})\left[\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{7}{2})}+2^2\ .\ \frac{1}{3!}\right]=14\sqrt{2\pi}.$$

Hence $x^{\frac{1}{2}}e^x=\sqrt{2\pi}[\frac{1}{2}J_{\frac{1}{2}}(x)+\frac{3}{2}J_{\frac{3}{2}}\cdot(x)+5J_{\frac{5}{2}}(x)+14J_{\frac{7}{2}}(x)+\ldots]$.

(*iv*) Let us now expand sin z. Let $k=1$.

Here $b_0=o$,

$b_1=2$,

$b_2=o$,

$b_3=8\ .\ 3\ .\ 2[\frac{1}{8}-\frac{1}{6}]=-2$,

$b_4=o$,

$$b_5=5\ !\left[\frac{32}{5\ !}-\frac{2}{2\ !\ 3\ !}-\frac{2}{3\ !\ 4}\right]=2.$$

Hence $\sin z=2J_1(z)-2J_3(z)+2J_5(z)-\ldots$.

5. The same method may be applied to expand any function $f(z)$ in the form

$$f(z)=\sum_{n=o}^{\infty} C_nJ_{na}(kz)$$

where a is an integer.

Now $$J_{na}(kz)=\left(\frac{k}{2}\right)^{na}\frac{z^{na}}{\Gamma(na+1)}\left[1-\left(\frac{k}{2}\right)^2\frac{z^2}{na+1}\right.$$

$$\left.+\left(\frac{k}{2}\right)^4\frac{z^4}{(na+1)(na+2)}-\ldots\right].$$

Hence $f(o)=C_0$,

$$f'(o)=o,\ f'''(o)=o,\ f^{v}(o)=o,\ \ldots$$

$$\frac{f''(o)}{2\,!}=-\left(\frac{k}{2}\right)^2\frac{C_0}{\Gamma(2)},\quad \frac{f^{iv}(o)}{4\,!}=\left(\frac{k}{2}\right)^4\frac{C_0}{\Gamma(3)},$$

etc.

$$\frac{f^{2r}(o)}{2r\,!}=\left(\frac{k}{2}\right)^{2r}\frac{(-)^r C_0}{\Gamma(r+1)}\text{ where }2r<a$$

If a be even,

$$\frac{f^{a}(o)}{a\,!}=\left(\frac{k}{2}\right)^{a}\frac{C_1}{\Gamma(a+1)}+(-1)^{\frac{a}{2}}\frac{C_0}{\Gamma(\frac{a}{2}+1)}\left(\frac{k}{2}\right)^{a};$$

$$f^{a+1}(o)=o,$$

$$\frac{f^{a+2}(o)}{(a+2)\,!}=-\left(\frac{k}{2}\right)^{a+2}\frac{C_1}{\Gamma(a+2)}+(-1)^{\frac{a}{2}+1}\frac{C_0}{\Gamma(\frac{a}{2}+2)}\left(\frac{k}{2}\right)^{a+2},$$

etc.

$$\frac{f^{2a}(o)}{2a\,!}=\left(\frac{k}{2}\right)^{2a}\frac{C_2}{\Gamma(2a+1)}+(-1)^{\frac{a}{2}}\left(\frac{k}{2}\right)^{2a}\frac{C_1}{\Gamma(a+\frac{a}{2}+1)}$$

$$+(-)^{a}\left(\frac{k}{2}\right)^{2a}\frac{C_0}{\Gamma(a+1)},$$

etc.

$$\frac{f^{na}(o)}{na\,!}=\left(\frac{k}{2}\right)^{na}\frac{C_n}{\Gamma(na+1)}+(-)^{\frac{a}{2}}\left(\frac{k}{2}\right)^{na}\frac{C_{n-1}}{\Gamma\{(n-1)a+\frac{a}{2}+1)\}}$$

$$+\ \ldots\ +(-)^{\frac{na}{2}}\left(\frac{k}{2}\right)^{na}\frac{C_0}{\Gamma(\frac{na}{2}+1)}.$$

Hence $C_0=f(o)$,

$$C_n=\left(\frac{2}{k}\right)^{na}\frac{f^{na}(o)}{na\,!}+(-)^{\frac{a}{2}+1}\frac{\Gamma(na+1)}{\Gamma\{(n-1)a+\frac{a}{2}+1)\}}C_{n-1}+\ldots$$

$$+(-)^{\frac{na}{2}}\frac{C_0\Gamma(na+1)}{\Gamma(\frac{na}{2}+1)}.$$

If a be odd,

$$f(o)=C_0,$$

$$\frac{f^{a}(o)}{a\,!}=\left(\frac{k}{2}\right)^{a}\frac{C_1}{\Gamma(a+1)},$$

$$\frac{f^{2a}(o)}{2a\,!}=\left(\frac{k}{2}\right)^{2a}\frac{C_2}{\Gamma(2a+1)}+(-)^{a}\frac{C_0}{\Gamma(a+1)}\left(\frac{k}{2}\right)^{2a},$$

$$\frac{f^{3a}(o)}{3a\,!}=\left(\frac{k}{2}\right)^{3a}\frac{C_3}{\Gamma(3a+1)}+(-)^{3a}\left(\frac{k}{2}\right)^{3a}\frac{C_1}{\Gamma(2a+1)},$$

etc.

Hence $C_0=f(o)$,

$$C_1=\left(\frac{2}{k}\right)^{a}f^{a}(o)$$

$$C_2=\left(\frac{2}{k}\right)^{2a}\Gamma(2a+1)\left\{\frac{f^{2a}(o)}{2a\,!}+(-)^{a+1}\left(\frac{k}{2}\right)^{2a}\frac{f(o)}{\Gamma(a+1)}\right\},$$

$$C_3=\left(\frac{2}{k}\right)^{3a}\Gamma(3a+1)\left\{\frac{f^{3a}(o)}{3a\,!}+(-)^{a+1}\left(\frac{k}{2}\right)^{2a}\frac{f^{a}}{\Gamma(2a+1)}\right\};$$

etc.

Some more examples are given below :—

(1) $x^{\frac{1}{3}}e^{x}=2^{\frac{1}{3}}\Gamma(\tfrac{1}{3})[\tfrac{1}{3}J_{\frac{1}{3}}(x)+\tfrac{8}{9}J_{\frac{4}{3}}(x)+\tfrac{77}{27}J_{\frac{7}{3}}(x)+\tfrac{1840}{243}J_{\frac{10}{3}}(x)+\ldots]$

(2) $x^{\frac{1}{4}}e^{x}=2^{\frac{1}{4}}\Gamma(\tfrac{1}{4})[\tfrac{1}{4}J_{\frac{1}{4}}(x)+\tfrac{5}{8}J_{\frac{5}{4}}(x)+\tfrac{63}{32}J_{\frac{9}{4}}(x)+\tfrac{325}{64}J_{\frac{13}{4}}(x)+\ldots]$

(3) $z^{\frac{1}{4}}\cos z=2^{\frac{1}{4}}\Gamma(\tfrac{1}{4})[\tfrac{1}{4}J_{\frac{1}{4}}(z)-\tfrac{27}{32}J_{\frac{9}{4}}(z)+\tfrac{595}{512}J_{\frac{17}{4}}(z)-\ldots]$

(4) $z^{\frac{1}{5}}\cos z=2^{\frac{1}{5}}\Gamma(\tfrac{1}{5})[\tfrac{1}{5}J_{\frac{1}{5}}(z)-\tfrac{77}{125}J_{\frac{11}{5}}(z)+\tfrac{2499}{3125}J_{\frac{21}{5}}(z)-\ldots]$.

A proof of the validity of the expansions given in this paper may be obtained by adopting the method given in Whittaker's *Modern Analysis*, Art. 17.81, pp. 368-369. The detail discussion of this question is reserved for a subsequent paper.

Before I conclude I should like to express my indebtedness to Prof. S. K. Banerjee at whose suggestion I took up and under whose guidance I carried on the above investigation.

A note on the theory of the magnetometer[1]

BY

L. SRIVASTAVA

[Read April 5th, 1920.]

The present paper contains the results of my investigations on magnetometric formulae (Gauss' A and B positions). In the usual Text Books the field round the *deflected needle* is taken uniform, *viz.*, the same at the centre as at the ends and the formulae are deduced. Here in this paper I have started with the inverse square law between the poles and thus got a system of forces which including Hm and Hm due to Earth's field are six in number. The formulae (5) and (5a) are the same as the usual Text Book formulae and it is found that, in order that these may form a very close approximation to actual relations, the deflections in A position should be 26·5° and in B position 31°.

In actual practice, however, it is found advantageous to use the sine position and not the Gauss' A and B positions and to use the empirical formulae

$$\frac{2mm'}{r^3}(1+\mathrm{P}r^{-2}+\mathrm{Q}r^{-4})$$

for the couple and

$$\frac{m}{\mathrm{H}}=\frac{r^3}{2}\sin u/(1+\mathrm{P}r^{-2}+\mathrm{Q}r^{-4}).$$

for the raito $\frac{m}{\mathrm{H}}$. P and Q are the distribution constants found by deflections for various values of r. Reference may be made to Lamont, *Proc. Roy. Soc.*, Vol. LXV, 1899.

Consider a small magnet A B free to rotate about its centre O in a horizontal plane, and let CD be the deflecting magnet (centre O′).

Let AB$=2a$, CD$=2c$, O′O$=x$ and $\angle$AON$=\theta$, N and S denote the north and south directions.

Also let $\pm m$ be the pole strengths of A, B and
„ „ $\pm m'$ „ „ „ C, D and let H be the horizontal intensity of the Earth's field.

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[1] Communicated by Prof. S. K. Banerjee.

There are six forces P, Q, R, S, Hm, Hm in the system (acting in the directions CA, AD, BC, DB, AN, BS. Now if p, q, r, s denote the perpendiculars from O on P, Q, R, S repectively, we get the condition of equilibrium by taking moments about O. Thus in Gauss' (A) position,

$$\mathrm{P}p-\mathrm{Q}q+\mathrm{R}r-\mathrm{S}s-\mathrm{H}m\cdot 2a\sin\theta=0. \qquad \ldots \quad (1)$$

But $\mathrm{P}\cdot p=\frac{mm'}{\mathrm{AC}^2}\cdot p=\frac{mm'}{\mathrm{AC}^3}\cdot p\cdot\mathrm{AC}=\frac{mm'}{\mathrm{AC}^3}2\Delta\mathrm{ADC}$

$$=\frac{mm'a\cdot\overline{x-c}\,\sin\overline{90+\theta}}{\{a^2+\overline{x-c}^2-a\cdot\overline{x-c}\,\cos\overline{90+\theta}\}^{\frac{3}{2}}}$$

$$=\frac{mm'\cdot a\cdot\overline{x-c}\,\cos\theta}{(a^2+\overline{x-c}^2+2a\cdot\overline{x-c}\,\sin\theta)^{\frac{3}{2}}}$$

$$=mm'\frac{d}{d\theta}\left\{\frac{1}{(a^2+\overline{x-c}^2+2a\cdot\overline{x-c}\,\sin\theta)^{\frac{1}{2}}}\right\}.$$

Similarly, by proper change in the signs of a and c, we get

$$\mathrm{Q}q=-mm'\frac{d}{d\theta}\left\{\frac{1}{(a^2+\overline{x+c}^2+2a\overline{x+c}\,\sin\theta)^{\frac{1}{2}}}\right\}.$$

Also $$\mathrm{R}r=mm'\frac{d}{d\theta}\left\{\frac{1}{(a^2+\overline{x-c}^2-2a\cdot\overline{x-c}\,\sin\theta)^{\frac{1}{2}}}\right\},$$

and

$$\mathrm{S}s=mm'\frac{d}{d\theta}\left\{\frac{1}{a^2+\overline{x+c}^2-2a\cdot\overline{x+c}\,\sin\theta)^{\frac{1}{2}}}\right\}.$$

Substituting these values of Pp, Qq, Rr, Ss in (1) we get,

$$mm'\frac{d}{d\theta}\left[\frac{-1}{(a^2+\overline{x-c}^2+2a\cdot\overline{x-c}\,\sin\theta)^{\frac{1}{2}}}+\frac{1}{(a^2+\overline{x+c}^2\,2a\cdot\overline{x+c}\,\sin\theta)^{\frac{1}{2}}}\right.$$

$$\left.+\frac{1}{(a^2+\overline{x-c}^2-2a\cdot\overline{x-c}\,\sin\theta)^{\frac{1}{2}}}-\frac{1}{(a^2+\overline{x+c}^2-2a\cdot\overline{x+c}\,\sin\theta)^{\frac{1}{2}}}\right]$$

$$=\mathrm{H}m\cdot 2a\sin\theta \qquad \ldots \quad (2)$$

Now putting $\frac{a}{x-c}=h, \frac{a}{x+c}=h'$ and $\sin\theta=\mu$, the left hand side expression in (2) becomes

$$mm'\frac{d}{d\theta}\left[\frac{-1}{\overline{x-c}|(2+2\mu h+h^2)^{\frac{1}{2}}}+\frac{1}{\overline{x+c}|(1+2\mu h'+h'^2)^1}\right.$$

$$\left.+\frac{1}{\overline{x-c}|(1-2\mu h+h^2)^{\frac{1}{2}}}-\frac{1}{\overline{x-c}|(1-2\mu h'+h'^2)^{\frac{1}{2}}}\right].$$

Expanding the radicals and putting

$$\frac{1}{(1-2\mu h+h^2)^{\frac{1}{2}}}=1+P_1h+P_2h^2+\ \dots\ +P_nh^n+\ \dots$$

where $P_1, P_2 \dots P_n$ are the Legendre's co-efficients (or Zonal Harmonics) and are functions of μ only. We get the left hand expression of (2)

$$=mm'\frac{d}{d\theta}\left[-\frac{1}{x+c}(1-P_1h+P_2h^2-P_3h^3+\dots)\right.$$

$$+\frac{1}{x+c}(1-P_1h'+P_2h'^2+\dots)+\frac{1}{x-c}(1+P_1h+P_2h^2+\dots)$$

$$\left.-\frac{1}{x+c}(1+P_1h'+P_2h'^2+\dots)\right]$$

$$=\frac{mm'd}{d\theta}\left[\frac{2}{x-c}\left\{P_1h+P_3h^3+\dots\right\}-\frac{2}{x+c}\left\{P_1h'+P_3h'^3+\dots\right\}\right]$$

$$=2mm'\frac{d}{d\theta}\left[P_1\left(\frac{h}{x-c}-\frac{h'}{x+c}\right)+P_3\left(\frac{h^3}{x-c}-\frac{h'^3}{x+c}\right)+\ \dots\ \right].$$

So equation (2) becomes

$$2mm'\frac{d}{d\theta}\left[P_1\left(\frac{h}{x-c}-\frac{h'}{x+c}\right)+P_3\left(\frac{h^3}{x+c}-\frac{h'^3}{x+c}\right)+\ \dots\ \right]$$

$$=Hm\cdot 2a\sin\theta.$$

Now putting for h and h' their values, and then dividing by $2am$, we have

$$m'\left[\left\{\frac{1}{(x-c)^2}-\frac{1}{(x+c)^2}\right\}\frac{dP_1}{d\theta}+\left\{\frac{a^2}{(x-c)^4}-\frac{a^2}{(x+c)^4}\right\}\frac{dP_3}{d\theta}+\ldots\right]=H\sin\theta\ \ldots\quad(3)$$

Now, if a be small compared with $(x-c)$ or $(x+c)$, the left hand series in (3) is rapidly convergent, and obviously the 1st term alone is a close approximation. Retaining the 1st term only, we get

$$m'\left[\left\{\frac{1}{(x-c)^2}-\frac{1}{(x+c)^2}\right\}\frac{dP_1}{d\theta}\right]=H\sin\theta\ \ldots\quad\ldots\quad(4)$$

but P_1 is easily seen to be equal to μ or $\sin\theta$ here. Therefore

$$\frac{dP_1}{d\theta}=\cos\theta.$$

Thus from (4)

$$m'\left[\left\{\frac{1}{(x-c)^2}-\frac{1}{(x+c)^2}\right\}\cos\theta\right]=H\sin\theta,$$

or

$$\frac{m'\cdot 4xc}{(x^2-c^2)^2}\cos\theta=H\sin\theta,$$

Putting $2m'c$ (magnetic moment of the magnet CD) $=$ M, we get

$$\frac{M}{H}=\frac{(x^2-c^2)^2}{2x}\tan\theta\ \ldots\quad\ldots\quad(5)$$

It is known that Legendre's Co-efficient are given by the formula

$$P_n=\frac{1}{2^n\cdot\lfloor\underline{n}}\left(\frac{d}{d\mu}\right)^n(\mu^2-1)^n.$$

Therefore,
$$P_3=\frac{1}{2^3\lfloor\underline{3}}\left(\frac{d}{d\mu}\right)^3(\mu^2-1)^3$$

$$=\frac{5\mu^3-3\mu}{2}.$$

Therefore, $\frac{dP_3}{d\theta}=\frac{dP_3}{d\mu}\cdot\frac{d\mu}{d\theta}=\frac{1}{2}\cos\theta\,[15\mu^2-3]$.

If then $\mu^2=\frac{1}{5}$, the second term of (3) including $\frac{dP_3}{d\theta}$ vanishes and the first term constitutes a still higher degree of approximation. But $\mu^2=\frac{1}{5}$ corresponds to $\sin^2\theta=\frac{1}{5}$ or $\sin\theta=\frac{1}{\sqrt{5}}=\sin 26^\circ{\cdot}5$ nearly, which gives $\theta=26^\circ{\cdot}5$.

This deflection then is a good one to obtain, when determing H by Gauss' (A) position.

Next we take Gauss' (B) position.

Here again we use the same terminology, and in addition, put, $OD=OC=(a^2+c^2)^{\frac{1}{2}}\equiv l$ say and $\frac{a}{l}=\sin\alpha$,and taking moments about O, we get

$$Pp-Qq+Rr-Ss+Hm\cdot 2a\sin\theta=0\ldots \qquad \ldots \quad (1a)$$

Now $Pp=\frac{mm'}{AC^2}\cdot p=\frac{mm'}{AC^3}\cdot p\cdot AC=\frac{mm'}{AC^3}\cdot 2\triangle AOC.$

$$=\frac{mm'al\sin(\pi-\overline{\theta-\alpha})}{[a^2+l^2-2al\cos(\pi-\overline{\theta-\alpha})]^{\frac{3}{2}}}=\frac{mm'al\sin(\theta-\alpha)}{\{a^2+l^2+2al\cos(\theta-\alpha)\}^{\frac{3}{2}}}$$

$$=mm'\frac{d}{d\theta}\left\{\frac{1}{(a^2+l^2+2al\cos\overline{\theta-\alpha})^{\frac{1}{2}}}\right\}$$

and by suitable alterations in the signs of a and α, we get

$$Qq=\frac{mm'al\sin(\theta+\alpha)}{[a^2+l^2+2al\cos(\theta+\alpha)]^{\frac{3}{2}}}=mm'\frac{d}{d\theta}\left\{\frac{1}{[a^2+l^2+2al\cos(\theta+\alpha)]^{\frac{1}{2}}}\right\}$$

$$Rr=\frac{mm'al\sin(\theta-\alpha)}{[a^2+l^2-2al\cos(\theta-\alpha)]^{\frac{3}{2}}}=mm'\frac{d}{d\theta}\left\{\frac{-1}{[a^2+l^2-2al\cos(\theta-\alpha)]^{\frac{1}{2}}}\right\}$$

and $$Ss=\frac{mm'al\sin(\theta+\alpha)}{[a^2+l^2-2al\cos(\theta+\alpha)]^{\frac{3}{2}}}=mm'\frac{d}{d\theta}\left\{\frac{-1}{[a^2+l^2-2al\cos(\theta+\alpha)]^{\frac{1}{2}}}\right\}.$$

Substituting the values of Pp, Qq, etc., in $(1\cdot a)$ we have

$$mm'\frac{d}{d\theta}\Bigg[\frac{1}{(a^2+l^2+2al\cos\overline{\theta-\alpha})^{\frac{1}{2}}}-\frac{1}{(a^2+l^2+2al\cos\overline{\theta+\alpha})^{\frac{1}{2}}}$$

$$-\frac{1}{\{a^2+l^2-2al\cos(\theta-\alpha)\}^{\frac{1}{2}}}+\frac{1}{(a^2+l^2+2al\cos\overline{\theta+\alpha})^{\frac{1}{2}}}\Bigg]$$

$$=-Hm\cdot 2a\sin\theta-\dots \qquad \dots \quad (2a)$$

Putting $\frac{a}{l}=h$, $\cos(\theta+\alpha)=\mu$, and $\cos(\theta-\alpha)=\mu'$, we get

$$mm'\frac{d}{d\theta}\Bigg[\frac{1}{l(1+2\mu'h+h^2)^{\frac{1}{2}}}-\frac{1}{l(1+2\mu h+h^2)^{\frac{1}{2}}}$$

$$-\frac{1}{l(1-2\mu h'+h^2)^{\frac{1}{2}}}+\frac{1}{l(1-2\mu h+h^2)^{\frac{1}{2}}}\Bigg]$$

$$=-Hm\cdot 2a\sin\theta.$$

or

$$\frac{mm'}{l}\frac{d}{d\theta}[(1-P'_1h+P'_2h^2-P'_3h^3+\dots)-(1-P_1h+P_2h^2-P_3h^3$$

$$+\dots)-(1+P'_1h+P'_2h^2+\dots)+(1+P_1h+P_2h^2+\dots)]$$

$$=-Hm.2a\sin\theta,$$

or

$$-\frac{mm'}{l}\frac{d}{d\theta}[2\,P'_1h+P'_3h^3+\dots)-2(P_1h+P_3h^3+\dots)]=Hm.2a\sin\theta$$

or

$$-\frac{2mm'}{l}\frac{d}{d\theta}[(P'_1-P_1)+(P'_1-P_3)h^2+\dots]=Hm\cdot 2a\sin\theta.$$

Now putting for h its value $\frac{a}{l}$, and dividing both sides by $-2am$, we get

$$\frac{m'}{l^2}\frac{d}{d\theta}\Big[(P'_1-P_1)+(P'_3-P_3)\frac{a^2}{l^2}+\dots\Big]=H\sin\theta\ \dots \qquad \dots \quad (3a)$$

Here again, if a is small compared with l, the series on the left hand side is rapidly convergent, and the first term alone constitutes a close approximation, thus

$$\frac{m'}{l^2}\frac{d}{d\theta}(P'_1-P_1)=H\sin\theta \;\ldots\qquad\ldots\quad (4\cdot a)$$

where obviously $P'_1=\mu'=\cos(\theta-a)$ and $P_1=\mu=\cos(\theta+a)$.

Hence $(4\cdot a)$ gives

$$\frac{m'}{l^2}(-\sin\overline{\theta-a}|+\sin\overline{\theta+a}|)=H\sin\theta,$$

or

$$\frac{m'}{l^2}\cdot 2\cos\theta\sin a=H\sin\theta,$$

or

$$\frac{m'}{l^2}\cdot 2\cos\theta\cdot\frac{c}{l}=H\sin\theta,$$

and if $M=2cm'$ (the magnetic moment of the magnet CD), we get

$$\frac{M}{H}=l^3\tan\theta=(a^2+c^2)^{\frac{3}{2}}\tan\theta \;\ldots.\qquad\ldots\quad (5\cdot a)$$

Here, also, if the second term of $(3\cdot a)$ is made to vanish, the 1st term will constitute a still higher degree of approximation. The 2nd term of $(3\cdot a)$ will vanish if

$$\frac{d}{d\theta}(P'_3-P_3)=0 \;\ldots\qquad\ldots\quad (6\cdot a)$$

where $P'_3=\dfrac{5\mu'^3-3\mu'}{2}$, and $P_3=\dfrac{5\mu^3-3\mu}{2}$ and $\mu'=\cos(\theta-a)$, $\mu=\cos(\theta+a)$.

Substituting the values of P'_3 and P_3 in $(6a)$ and differentiating we get,

$$(5\mu'^2-1)\frac{d\mu'}{d\theta}-(5\mu^2-1)\frac{d\mu}{d\theta}=0,$$

or

$$-[5\cos^2(\theta-a)-1]\sin(\theta-a)-[5\cos^2(\theta+a)-1]\sin(\theta+a)=0,$$

and on simplifying we get

$$[\sin(\theta-\alpha)-\sin(\theta+\alpha)][5\{\sin^2(\theta-\alpha)+\sin(\theta-\alpha)\sin(\theta+\alpha)+\sin^2(\theta+\alpha)\}-4]=0,$$

which gives either $\sin(\theta-\alpha)-\sin(\theta+\alpha)=0$,

[*i.e.*, $\alpha=0$ or $\theta=90$ which is not practical]

or $$5\{\sin^2(\theta-\alpha)+\sin(\theta-\alpha)\sin(\theta+\alpha)+\sin^2(\theta+\alpha)\}-4=0,$$

that is, $$\sin^2\theta[3-4\sin^2\alpha]=[\tfrac{4}{5}-\sin^2\alpha],$$

or $$\sin^2\theta=\frac{\frac{4}{5}-\sin^2\alpha}{3-4\sin^2\alpha}=\frac{4}{15}$$ nearly when α is small

Therefore, we get $\sin\theta=\dfrac{2}{\sqrt{15}}=\sin 31^\circ$ nearly.

Thus $\theta=31^\circ$, and this deflection then is a good one to obtain when finding H by Gauss' B position.

On the motion of an Elongated Spheroid in Viscous Fluid Media

By

BHOLANATH PAL.

[*Read November 16th, 1919*]

It is well known that the problem of the motion of a sphere in a viscous fluid medium has been solved by various investigators. But the corresponding problem for a spheroid or an ellipsoid has been attempted with success by a few. In a previous paper[1], I investigated the motion of a spheroid of small ellipticity, *i.e.*, a spheroid slightly differing from a sphere, in a viscous fluid. In the present paper, I have used the same method to investigate the other extreme case, viz., the motion of a very elongated spheroid in a viscous liquid along its axis.

It is easy to see that an equation[2] of the type

$$r=a\{1+2(1-\epsilon)\ P_2\ (\cos\theta)\}$$

will represent approximately an elongated spheroid or very roughly a finite rod of length $(3-2\epsilon)\ a$, provided ϵ is an extremely small quantity, whose square and higher powers are negligible.

Let u, v, w be the three velocity-components and suppose $\mathrm{u}=u+\mathrm{U}$, where U is the velocity of the spheroid.

The differential equations for the motion of the fluid are

$$\mathrm{U}\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u,$$

$$\mathrm{U}\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v,$$

$$\mathrm{U}\frac{\partial w}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w,$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0;$$

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[1] *Bulletin of the Cal Math. Soc.*, Vol. X, No. 2, Sept, (1919).

[2] This method of representing an elongated spheroid was suggested to me by Prof. Banerji. He is using this equation to solve a number of interesting problems involving an elongated spheroid or a thin finite rod, for example, the problem of the electrical oscillations on an elongated spheroid or a thin rod, the radiation from the two ends of a finite rod, etc.

where p is the pressure, ρ the density, and ν the kinematic co-efficient of viscosity.

The corresponding auxiliary conditions are $u=0$, $v=0$, $w=0$, at infinity, and $u=-\mathrm{U}$, $v=0$, $w=0$, on the surface.

From the equations of motion, we easily obtain

$$\nabla^2 p=0.$$

Assuming $$p=\rho \mathrm{U}\frac{\partial \phi}{\partial x}, \text{ where } \nabla^2\phi=0,$$

we take

$$u=-\frac{\partial \phi}{\partial x}+u',$$

$$v=-\frac{\partial \phi}{\partial y}+v',$$

$$w=-\frac{\partial \phi}{\partial z}+w'.$$

Substituting this value of u in the first equation of the equations of motion we get

$$\left(\nabla^2-\frac{\mathrm{U}}{\nu}\frac{\partial}{\partial x}\right)u'=0.$$

Writing k for $\frac{\mathrm{U}}{2\nu}$, this can be written in the from

$$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)u'=0.$$

Similarly $$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)v'=0,$$

$$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)w'=0,$$

and $$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0.$$

The vortex lines will be circles having the axis of x as a common axis ; we may, therefore, assume

$$\xi=0, \quad \eta=-\frac{\partial \mathrm{X}}{\partial z}, \quad \zeta=\frac{\partial \mathrm{X}}{\partial y},$$

where $\mathbf{X}$ is a function of x and ω (the distance from the axis of x) only and ξ, η, ζ are components of vorticity.

Then X must satisfy the equation

$$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)X=0.$$

Thus

$$2k\frac{\partial u'}{\partial x}=\frac{\partial^2 X}{\partial x^2}-2k\frac{\partial X}{\partial x},$$

$$2k\frac{\partial v'}{\partial x}=\frac{\partial^2 X}{\partial x\partial y},$$

$$2k\frac{\partial w'}{\partial x}=\frac{\partial^2 X}{\partial x\partial z}.$$

Therefore

$$u'=\frac{1}{2k}\frac{\partial X}{\partial x}-X,$$

$$v'=\frac{1}{2k}\frac{\partial X}{\partial y},$$

$$w'=\frac{1}{2k}\frac{\partial X}{\partial z}.$$

In the differential equation

$$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)X=0,$$

writing $e^{kx}X'$ for X, we obtain

$$(\nabla^2-k^2)X'=0$$

The solution[1] of this differential equation is

$$X'=\sum_{0}^{\infty} C'_n f_n(kr)r^n P_n$$

where

$$f_n(\zeta)=\left(\frac{d}{\zeta d\zeta}\right)^n\frac{e^{-\zeta}}{\zeta}$$

i.e.

$$f_n(\zeta)=(-1)^n\frac{e^{-\zeta}}{\zeta^{n+1}}\left[1+\frac{n(n+1)}{2\cdot\zeta}+\frac{(n-1)n(n+1)(n+2)}{2\cdot4\cdot\zeta^2}+\ldots\right.$$

$$\left.\ldots+\frac{1\cdot2\cdot3\ldots2n}{2\cdot4\cdot6\ldots2n\cdot\zeta^n}\right],$$

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[1] See Prof. Lamb's Hydrodynamics, Third edition, p. 480.

Thus

$$f_o(\zeta)=\frac{e^{-\zeta}}{\zeta}\ ,f_1(\zeta)=-\left(\frac{1}{\zeta^2}\,2+\frac{1}{\zeta^3}\right)e^{-\zeta}\ ,f_2(\zeta)=\left(\frac{1}{\zeta^3}+\frac{3}{\zeta^4}+\frac{3}{\zeta^5}\right)e^{-\zeta},$$

etc.

Therefore

$$X'=\left[\ C_o+C_1\left\{1+\frac{\alpha}{r}\right\}P_1+C_2\left\{1+\frac{\beta}{r}+\frac{\gamma}{r^2}\right\}2P_2+\ldots\right]\frac{e^{-kr}}{r}$$

where α, β, γ...are all known constants such that

$$\alpha=\frac{1}{k}\ ,\ \beta=\frac{3}{k}\ ,\ \gamma=\frac{3}{k^2}\ \ldots$$

From the foregoing relation we get

$$X=\left[\ C_o+C_1\left\{1+\frac{\alpha}{r}\right\}P_1+C_2\left\{1+\frac{\beta}{r}+\frac{\gamma}{r^2}\right\}2P_2+\ldots\right]\frac{e^{-k(r-x)}}{r}.$$

Taking kr very small and neglecting square of the velocity this can be written in the from

$$X=\left[\ C_o+C_1\left\{1+\frac{\alpha}{r}\right\}P_1+C_2\left\{1+\frac{\beta}{r}+\frac{\gamma}{r^2}\right\}2P_2+\ldots\right]\left\{\frac{1}{r}-k+\frac{kx}{r}+\ldots\right\}.$$

From the auxiliary conditions we can easily find

$$\phi=\frac{A_o}{r}+A_1\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+A_2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ldots$$

where A_0, A_1, A_2...are different constants.

We have

$$u=-\frac{\partial\phi}{\partial x}+\frac{1}{2k}\frac{\partial X}{\partial x}-X,$$

$$v=-\frac{\partial\phi}{\partial y}+\frac{1}{2k}\frac{\partial X}{\partial y},$$

$$w=-\frac{\partial\phi}{\partial z}+\frac{1}{2k}\frac{\partial X}{\partial z}.$$

Thus finally we obtain

$$\begin{aligned}
u=&-\frac{\partial}{\partial x}\left\{\frac{A_0}{r}+A_1\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+A_2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\dots\right\}\\
&-\frac{C_0}{2k}\left\{\frac{4}{3}\frac{k}{r}-\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\frac{1}{3}kr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\dots\right\}\\
&-\frac{C_1}{2k}\left\{\frac{2k}{3r}-\frac{1}{3r^2}-\frac{6kr}{5}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)-\frac{1}{3}kr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right\}\\
&\qquad\qquad+\frac{2}{3}r\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)-\frac{2}{3\cdot5}kr^3\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)\Big\}\\
&-\frac{aC_1}{2k}\left\{\frac{k}{3r^2}-\frac{9}{5}k\frac{\partial}{\partial x}\left(\frac{1}{r}\right)-\frac{2}{3}kr\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right.\\
&\qquad\qquad\left.-\frac{1}{5}kr^2\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)\right\}\\
&+\frac{C_2}{2k}\left\{\frac{8}{15}\frac{k}{r}-\frac{8}{5}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)+\frac{12}{5}kr\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right.\\
&-\frac{26}{21}kr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\frac{3}{5}r^2\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)-\frac{2}{5}kr^3\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)\\
&\qquad\qquad\left.-\frac{3}{5\cdot7}kr^4\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right\}\\
&+\frac{\beta C_2}{2k}\left\{\frac{4}{15}\frac{k}{r^2}-\frac{4}{5}\frac{1}{r}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\frac{8}{5}k\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right.\\
&-\frac{37}{21}kr\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\frac{4}{5}r\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)-\frac{3}{5}kr^2\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)\\
&\qquad\qquad\left.-\frac{4}{5\cdot7}kr^3\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right\}\\
&+\frac{\gamma C_2}{2k}\left\{\frac{4}{5}\frac{k}{r}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)-\frac{16}{7}k\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)\right.\\
&\qquad\qquad\left.-\frac{4}{5}kr\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)-\frac{1}{7}kr^2\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right\}+\dots
\end{aligned}$$

and similar two expressions for v and w.

The constants A_0. A_1, A_2...and C_0, C_1 C_2...are to be determined from the auxiliary conditions.

We have $u=-U$, when $r=a\{1+2(1-\epsilon)P_2\}$.

Therefore

$$-U=\frac{A_0}{3^3a^2}\left\{\frac{3(1+2\epsilon)}{\mu^3}-\frac{2\epsilon}{\mu^5}\right\}-\frac{A_1}{3^4a^3}\left\{\frac{9(1+3\epsilon)}{\mu^4}-\frac{3(1+6\epsilon)}{\mu^6}+\frac{3\epsilon}{\mu^8}\right\}$$

$$+\frac{3A_2}{3^5a^4}\left\{\frac{15(1+4\epsilon)}{\mu^5}-\frac{(9+56\epsilon)}{\mu^7}+\frac{12\epsilon}{\mu^9}\right\}-\frac{2C_0}{3^3a}\left\{\frac{3(1+\epsilon)}{\mu^2}-\frac{\epsilon}{\mu^4}\right\}$$

$$-\frac{C_0}{3^3 2ka^2}\left\{\frac{3(1+2\epsilon)}{\mu^3}-\frac{2\epsilon}{\mu^5}\right\}-\frac{C_0}{2.3^3a}\left\{9(1+\epsilon)-\frac{3(1+2\epsilon)}{\mu^2}+\frac{\epsilon}{\mu^4}\right\}+\dots$$

$$-\frac{C_1}{3^3a}\left\{\frac{3(1+\epsilon)}{\mu^2}-\frac{\epsilon}{\mu^4}\right\}+\frac{C_1}{2k\cdot 3^4a^2}\left\{\frac{3(1+2\epsilon)}{\mu^4}-\frac{2\epsilon}{\mu^6}\right\}$$

$$-\frac{C_1}{5.3a}\left\{\frac{3(1+\epsilon)}{\mu}-\frac{\epsilon}{\mu^3}\right\}$$

$$+\frac{C_1}{2.3^3a}\left\{9(1+\epsilon)-\frac{3(1+2\epsilon)}{\mu^2}+\frac{\epsilon}{\mu^4}\right\}-\frac{C_1}{k.3^4a^2}\left\{\frac{9(1+2\epsilon)}{\mu^2}\right.$$

$$\left.-\frac{3(1+4\epsilon)}{\mu^4}+\frac{2\epsilon}{\mu^6}\right\}$$

$$-\frac{C_1}{5.3^2a}\left\{15(1+\epsilon)\mu-\frac{(9+14\epsilon)}{\mu}+\frac{3\epsilon}{\mu^3}\right\}+\dots$$

$$-\frac{aC_1}{2.3^4a^2}\left\{\frac{3(1+2\epsilon)}{\mu^4}-\frac{2\epsilon}{\mu^6}\right\}-\frac{aC_1}{2.5.3a^2}\left\{\frac{3(1+2\epsilon)}{\mu^3}-\frac{2\epsilon}{\mu^5}\right\}$$

$$+\frac{aC_1}{3^4a^3}\left\{\frac{9(1+2\epsilon)}{\mu^2}-\frac{3(1+4\epsilon)}{\mu^4}+\frac{2\epsilon}{\mu^6}\right\}-\frac{aC_1}{2k.3^4a^3}\left\{\frac{9(1+3\epsilon)}{\mu^4}\right.$$

$$\left.-\frac{3(1+6\epsilon)}{\mu^6}+\frac{3\epsilon}{\mu^8}\right\}$$

$$-\frac{aC_1}{2.5.3^2a^2}\left\{\frac{15(1+2\epsilon)}{\mu}-\frac{(9+28\epsilon)}{\mu^3}+\frac{6\epsilon}{\mu^5}\right\}-\dots$$

$$+\frac{4C_2}{5.3^3a}\left\{\frac{3(1+\epsilon)}{\mu^2}-\frac{\epsilon}{\mu^4}\right\}+\frac{4C_2}{5k.3^3a^2}\left\{\frac{3(1+2\epsilon)}{\mu^3}-\frac{2\epsilon}{\mu^5}\right\}$$

$$-\frac{2C_2}{5.3a}\left\{\frac{3(1+\epsilon)}{\mu}-\frac{\epsilon}{\mu^3}\right\}$$

$$-\frac{13C_2}{7.3^3a}\left\{9(1+\epsilon)-\frac{3(1+2\epsilon)}{\mu^2}+\frac{\epsilon}{\mu^4}\right\}-\frac{C_2}{2k.5.3a^2}\left\{\frac{15(1+2\epsilon)}{\mu}\right.$$

$$\left.-\frac{(9+28\epsilon)}{\mu^3}+\frac{6\epsilon}{\mu^5}\right\}$$

$$+\frac{\mathrm{C}_2}{5.3a}\left\{15(1+\epsilon)\mu-\frac{(9+14\epsilon)}{\mu}+\frac{3\epsilon}{\mu^3}\right\}$$

$$-\frac{\mathrm{C}_2}{5.7a}\left\{105(1+\epsilon)\mu^2-(90+125\epsilon)+\frac{(9+39\epsilon)}{\mu^2}-\frac{3\epsilon}{\mu^4}\right\}+\dots$$

$$+\frac{2\beta \mathrm{C}_2}{5.3^4a^2}\left\{\frac{3(1+2\epsilon)}{\mu^4}-\frac{2\epsilon}{\mu^6}\right\}+\frac{2\beta \mathrm{C}_2}{5k.3^4a^3}\left\{\frac{3(1+3\epsilon)}{\mu^5}-\frac{3\epsilon}{\mu^7}\right\}$$

$$-\frac{4\beta \mathrm{C}_2}{5.3^3a^2}\left\{\frac{3(1+2\epsilon)}{\mu^3}-\frac{2\epsilon}{\mu^5}\right\}-\frac{37\beta \mathrm{C}_2}{14.3^4a^2}\left\{\frac{9(1+2\epsilon)}{\mu^2}-\frac{(3+12\epsilon)}{\mu^4}+\frac{2\epsilon}{\mu^6}\right\}$$

$$-\frac{2\beta \mathrm{C}_2}{5k\cdot 3^3a^3}\left\{\frac{15(1+3\epsilon)}{\mu^3}-\frac{(9+42\epsilon)}{\mu^5}+\frac{9\epsilon}{\mu^7}\right\}$$

$$+\frac{\beta \mathrm{C}_2}{10.3a^2}\left\{\frac{15(1+2\epsilon)}{\mu}-\frac{(9+28\epsilon)}{\mu^3}+\frac{6\epsilon}{\mu^5}\right\}$$

$$-\frac{2\beta k}{5.7.3^2a^2}\left\{104(1+2\epsilon)-\frac{(90+250\epsilon)}{\mu^2}+\frac{(9+78\epsilon)}{\mu^4}-\frac{6\epsilon}{\mu^6}\right\}+\dots$$

when μ is written for cos θ.

Now this must hold for all values of cos θ. Therefore equating to zero the co-efficients of various powers of $\frac{1}{\mu}$, we get a series of equations from which the values of the constants C_0, C_1, C_2,...and A_0, A_1, A_2,...can be determined.

Thus we have

$$\mathrm{U}=\frac{(1+\epsilon)}{6a}\mathrm{C}_0-\frac{(1+\epsilon)}{6a}\mathrm{C}_1-\frac{\{(41a-14\beta)+(62a-48\beta)\epsilon\}}{21a^2}\mathrm{C}_2+\dots$$

$$0=\frac{(1+\epsilon)}{3a}\mathrm{C}_1-\frac{(1+\epsilon)}{a}\mathrm{C}_2+\dots$$

$$0=\frac{(3+2\epsilon)}{2.3^2a}\mathrm{C}_0+\frac{(3+4\epsilon)}{2.3^2a}\mathrm{C}_1-\frac{\{(24a+175\beta)-(386a-630\beta)\epsilon\}}{5.14.3^2a^2}\mathrm{C}_2+\dots$$

$$0=\frac{(1+2\epsilon)}{3^2a^2}\mathrm{A}_0-\frac{(1+2\epsilon)}{2k\cdot 3^2a^2}\mathrm{C}_0-\frac{\epsilon}{k\cdot 3^2a^2}\mathrm{C}_1-\frac{\{7+(20-3ka)\epsilon\}}{k\cdot 3^2a^2}\mathrm{C}_2+\dots$$

$$0=\frac{(1+3\epsilon)}{3^2a^3}\,A_1-\frac{\epsilon}{2.3^2a}\,C_0-\frac{\epsilon}{2.3^2a}\,C_1-\frac{\{105-(140-8ka)\epsilon\}}{2k.35.3^2a^2}\,C_2+\ldots$$

$$0=\frac{2\epsilon}{3^3a^2}A_0-\frac{5(1+4\epsilon)}{3^3a^4}\,A_2-\frac{\epsilon}{k.3^3a^2}C_0-\frac{14\epsilon}{k.3^3a^2}\,C_2+\ldots$$

... ...

... ...

Keeping C_0, C_1, C_2 and A_0, A_1, A_2 only and neglecting the square of the velocity the following approximate values of the constants can be easily obtained.

$$C_0=\frac{97}{102}\left\{1+\frac{15}{34}\epsilon\right\}Ua,$$

$$C_1=-\frac{35}{34}\left\{1-\frac{57}{34}\epsilon\right\}Ua,$$

$$C_2=-\frac{35}{102}\left\{1-\frac{57}{34}\epsilon\right\}Ua,$$

... ...

... ...

and

$$A_0=-\frac{131}{34}\left\{1+\frac{117}{34}\epsilon\right\}va,$$

$$A_1=-\frac{4\epsilon}{52}\,Ua^3,$$

$$A_2=0.$$

... ...

... ...

Thus ϕ and X can be found and u, v, w can be determined in terms of these two functions.

I am grateful to Prof. S. K. Banerjee for his valuable suggestion and encourgement.

Review

CULLIS'S MATRICES AND DETERMINOIDS.

[*Continued from Vol. X, No. 4.*]

THE MATRIX.[1]

PHILONOUS—I am not for imposing any sense on your words: you are at liberty to explain them as you please. Only I beseech you make me understand something by them.

BERKELEY,
The First dialogue between Hylas and Philonous.

With Chapter VI, the theory *proper* of matrices begins. Hitherto the matrix has been considered only as a rectangular *arrangement* of the scalar numbers and the determinoid only has played a considerable part. Square matrices, unlike " determinoids " has a previous history which is summed up below, in the inimitable style of Sylvester to whom and to Cayley, in its early stages, the theory of matrices owes so much for its development:—

A historical retrospect.

" A matrix of a *quadrate* form *historically* takes its rise in the notion of a linear substitution upon a system of variables or carriers ; regarded apart from the *determinant* which it may be and at one time was almost exclusively used to represent, it becomes an empty schema of operation, but in conformity with Hegel's principle that the Negative is the course through which thought arrives at another and a fuller positive, only for a moment loses the attribute of *quantity,* if it be allowed that that term is properly applied to whatever is the subject of *functional operation,* of a *higher* and *unthought of kind,* and so to say, in a glorious shape—as an organism composed of discrete parts but having an essential and undivisible unity as a whole of its own. *Naturum expellus furca tamen usque recurret.* The conception of Multiple quantity thus rises upon the field of vision."

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[1] Sir Thomas Muir gave in Vol 20 of the American Journal of Mathematics, a list of writings on the Theory of Matrices from 1857 to 1893. The Bibliography at the end of Shaw's Synopsis of Linear Associative Algebra brings the list down to 1905.

"At first undifferentiated from their content, matrices come to be regarded as susceptible of being multiplied together ; the word multiplication strictly applicable at that stage of evolution to the *content* alone, getting transferred by a fortunate confusion of language to the schema and superseding, to some extent, the use of the more appropriate word, composition, applied to the reiteration of substitution in the Theory of Numbers. Thus, there comes into view a process of multiplication which the mind, almost at a glance, is able to recognize, must be subject to the Assosciative law of ordinary multiplication, although not so to the Commutative law ; but the full significance of this fact lay hidden until the subject-matter of such operations had dropped its provisional mantle, its aspect as a mere schema and stood revealed as *bonâfide* multiple Quantity subject to all the affections and lending itself to all the operations of ordinary numerical quantity. This revolution [1] was effected by a forcible injection into the subject of the concept of *addition*, *i.e.*, by choosing to regard matrices as susceptible of being *added* to one another ; a notion as it seems to me quite foreign to the idea of substitution, the *Nidus* in which that of multiple quantity was laid, hatched and reared. This step was as far as I know, first made by Cayley in his memoir on matrices (1858, Vol. II, Collected Works) wherein he may be said to have laid the foundation-stone of the science of Multiple Quantity (*N.B.* But Willard Gibbs contended, and rightly I think, that the key to the theory of matrices is given in the first Ausdehnungslehre, *i.e.*, of 1844 ; *vide* page 167, Vol. II, of the scientific papers of Gibbs). That memoir indeed (it seems to me) may, with truth, be affirmed to have ushered in the reign of Algebra the Second ; just as Algebra the First, in its character, not as mere art or mystery, but as science and philosophy took its rise in Harriot's *Artis Analyticae Praxis*, published in 1631, ten years after his death, and exactly 250 years before I gave the first course of lectures ever delivered on Multinomial Quantity, in 1881, at the Johns Hopkins University. Much as I owe in the way of fruitful suggestion to Cayley's immortal memoir, the idea of subjecting matrices to the *additive* process and of their consequent amenability to the laws of functional operation was not taken from it but occurred to me independently before I had seen the memoir or was acquainted with its contents ; and indeed forced itself on my attention as a means of giving simplicity and generality to my formula for the powers or roots of matrices. My memoir on Tchebycheff's method concerning the totality of prime numbers within certain limits, was *the indirect cause of turning my attention* to the subject. All this only proves how far the discovery of the quantitative nature of matrices is removed from being *artificial* or *factitious*, but, on the contrary, was bound to be evolved, in the fulness of time, as a necessary sequence to previously acquired cognitions."

---

[1] The late Maxime Bocher said " The Matrix first appears as *inanimate* object, a mere rectangular *array* of quantities from which determinants whose vanishing and non-vanishing are observed or whose values are used are cut out. Later, however, the matrix appears as an *animated being* capable of combination by *addition* and at least two kinds of multiplication with its kind and it is in *this aspect that the conception becomes a highly fruitful one*."

" Already in Quaternions the example had been given of Algebra released from the yoke of the commutative principle of multiplication —an emancipation somewhat akin to Lobatchewsky's of Geometry from Euclid's noted empirical axiom, and later on the Pierces, father and son (but subsequently to 1858) had prefigured the universalization of Hamilton's theory and had emitted an opinion to the effect that probably all systems of algebraical symbols subject to the Associative law of multiplication, would be eventually found to be identical with linear transformations of schemata susceptible of matricular representation."

" That such must be the case it would be rash to assert ; but it is very difficult to conceive how the contrary can be true or where to seek, outside the concept, of *substitution* for matter affording pabulum to the principle of free consosciation of successive actions or operations."

Cayley in his " Memoir on the Theory of Matrices " (Vol. II of his Collected Mathematical Works—pages 475-496) writes the matrix of the third order as

$$\begin{matrix}(a & b & c)\\ |a' & b' & c'| \\ |a'' & b'' & c''|\end{matrix}$$

" The notion of such a matrix " he says " arises naturally from an abbreviated notation for a set of *linear* equations, *viz.*, the equations

$$X=ax+by+cz,$$

$$Y=a'x+b'y+c'z,$$

$$Z=a''x+b''y+c''z,$$

may be more simply represented by

$$(X, Y, Z)=\begin{array}{lcl} (a, & b, & c)(x, y, z) \\ \vert\, a' & b' & c' \,\vert \\ \vert\, a'' & b'' & c'' \,\vert \end{array}$$

and the consideration of such a system of equations leads to most of the fundamental notions in the theory of matrices."

Writing the equations

$$(X, Y, Z)=\begin{array}{lcl} (a, & b, & c)(x, y, z) \\ \vert\, a' & b' & c' \,\vert \\ \vert\, a'' & b'' & c'' \,\vert \end{array}$$

and

$$(x, y, z)=\begin{array}{lcl} (\alpha, & \beta, & \gamma)(\xi, \eta, \zeta) \\ \vert\, \alpha' & \beta' & \gamma' \,\vert \\ \vert\, \alpha'' & \beta'' & \gamma'' \,\vert \end{array}$$

Cayley obtains

$$(X, Y, Z)=\begin{array}{lcl} (A, & B, & C)(\xi, \eta, \zeta) \\ \vert\, A' & B' & C' \,\vert \\ \vert\, A'' & B'' & C'' \,\vert \end{array}$$

$$\begin{array}{lcllcl} (a, & b, & c)(\alpha, & \beta, & \gamma)(\xi, \eta, \zeta) \\ = \vert\, a' & b' & c' \,\vert & \vert\, \alpha' & \beta' & \gamma' \,\vert \\ \;\;\;\vert\, a'' & b'' & c'' \,\vert & \vert\, \alpha'' & \beta'' & \gamma'' \,\vert \end{array}$$

From which substituting for the matrix

$$\begin{array}{lcl} (A, & B, & C) \\ \vert\, A' & B' & C' \,\vert \\ \vert\, A'' & B'' & C'' \,\vert \end{array}$$

its value, obtains the well known rule for the multiplication or composition of two matrices.

In his celebrated memoir Cayley restricts his investigation mainly to *square* matrices. It is shown that matrices (attending to those only of the same order) "comport themselves as single quantities ;

they may be *added, multiplied* or compounded together, *etc.* The law of the addition of matrices is precisely similar to that for the addition of ordinary algebraical quantities; as regards their multiplication (or composition) there is the peculiarity that matrices are not in general *convertible*; it is nevertheless possible to form the powers (positive or negative, integral or fractional) of a matrix and thence to arrive at the notion of a rational and integral function or generally of any algebraic function, of a matrix. Cayley obtains the remarkable theorem that *any matrix whatever satisfies an algebraic equation of its own order,* the co-efficient of the highest power being unity, those of the other powers being functions of the terms of the matrix, the last co-efficient being in fact the determinant. This theorem shows that every rational and integral function (or indeed every rational functional) of a matrix may be considered as a rational and integral function, the degree of which is at most equal to that of the matrix less unity; it even shows that in a sense the same is true of any algebraic function whatever of a matrix. One of the applications of the theorem is the *finding of the general expression of matrices which are convertible with a given matrix.* The theory of rectangular matrices appeared to Cayley much less important than that of a square matrices and he devotes only a few articles to them at the end of his memoir.

As the very general theorem referred to above formed the basis of subsequent developments of the Matrix Theory it may be interesting to mention it here :—

The Identical Equation.

Let M be any matrix $[a]_m^m$. Then the derived determinant

$$\begin{vmatrix} a_{11}-\mathrm{M}, & a_{12} \;\ldots & a_{1m} \\ a_{21} & a_{22}-\mathrm{M}\ldots & a_{2m} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ a_{m1}\ldots & . & a_{mm}-\mathrm{M} \end{vmatrix}$$

vanishes considered as a *matrix* equation of the m-th order. Cayley writes the Identical equation, symbolically, as Det ($\bar{1}$. M $-\bar{\mathrm{M}}$. 1)=0 where the matrix M considered as a single entity (quantity) is represented by $\bar{\mathrm{M}}$ and taking 1 to denote the matrix-unity, $\bar{1}$. M would represent the matrix M and M.1 would represent the matrix M considered as a single quantity *involving the matrix unity.*

Cayley gives the proof of the Identical equation in the case of the matrix of the second order and gives only the enunciation in the case of the third order. An elementary proof in the case of the matrix of the third order (which is applicable also to the general case of any order) is given by Forsyth in the Messenger of Mathematics, Vol. 13 as well as by Muir in his article on Bipartitite Functions and Determinants in the Proceedings of the London Mathematical Society, Vol. 16. Arthur Bucheim's elegant paper on the "Theory of Matrices" (Vol. 16 of the Proceedings of the London Mathematical Society) connects Grassmann's methods (in which the key to the matrix theory lay) with the theory created by Cayley and Sylvester, by connecting them both with Hamilton's investigations. Bucheim shows that the linear and vector function of a vector is simply the matrix of the third order.

A brief survey, which is here attempted of the previous history of the Matrix Theory some account of which is necessary in order that we may be able to make an estimate of the remarkably original work of Dr. Cullis, will be incomplete if we did not refer to the writings of W. H. Metzler and Henry Taber. Metzler wrote on "the roots of matrices" (Vol XIV of the American Journal) and on "certain properties of symmetric, skew-symmetric and orthogonal matrices (Vol XV, *ibid*). His paper on the function ϕ (n, m) has been already noticed above. Taber's most important paper on the theory of matrices is that contained in Vol. 12 of the American Journal of Mathematics. According to him " Hamilton must be regarded as the originator of the theory of matrices, as he was the first to show that the symbol of a linear transformation might be made the subject matter of a calculus. Cayley makes no reference to Hamilton (in his Memoir of 1858) and was (probably) unaware that results essentially identical with some of his, had been obtained by Hamilton (or was it because Quaternions did not appeal to him as it did not to Lord Kelvin !!). On the other hand, Hamilton's results related only to matrices of the third and fourth order while Cayley's method was absolutely general. The identity of the two theories was explicitly mentioned by Clifford in a passage of his Dynamic (Clifford's writings on the subject are contained in Vols. I, II and III of his Dynamic, page 162 etseqelae and also in his Collected Papers, pp. 337-341).

Linear Associative algebras.

Subsequent to Cayley but previous to Sylvester, the Pierces, especially Charles Pierce, were led to the consideration of matrices from a *different point of view ; viz.*, from the investigation of Linear Associative Algebras, involving any number of linearly *independent units.* In this aspect the quantities arranged in a square are looked upon as the scalar co-efficients of the several units or "vids" of an algebra belonging to a certain class. Charles Pierce did something more. In his remarkable investigations upon the extension of the Boolian Calculus (*vide* Boole's Laws of Thought recently reprinted by the Open Court Publishing Company) to the Logic of relatives (*vide* Americal Journal of Mathematics, Vol. IV, page 221 and the memoirs of the American Academy of Sciences for 1870 and 1875) he came upon a set of forms (the "vids" mentioned above), constituting a system virtually identical with the calculus of matrices. He made the discovery that the whole theory of Linear Associative Algebra is included in the Theory of matrices ; in particular, he gave the matrix form of all the algebras considered by his father Benjamin Pierce, in his Linear Associative Algebra (Americal Journal, Vol. IV). A simple example may be given here to show how a *matric* algebra may be viewed as a *linear* algebra :—

Consider, to fix ideas, the two matrices of the 2nd order

$$m=\begin{bmatrix} ab \\ cd \end{bmatrix} \quad \mu=\begin{bmatrix} \alpha\beta \\ \gamma\delta \end{bmatrix}$$

and the four matrices

$$e_{11}=\begin{bmatrix} 10 \\ 00 \end{bmatrix} \; e_{12}=\begin{bmatrix} 01 \\ 00 \end{bmatrix}$$

$$e_{21}=\begin{bmatrix} 00 \\ 10 \end{bmatrix} \; e_{22}=\begin{bmatrix} 00 \\ 01 \end{bmatrix}$$

The sixteen products of these four by two's are

$$\left.\begin{array}{l} e_{ij}e_{jk}=e_{ik} \\ e_{ij}e_{tk}=0(t\neq j) \end{array}\right\} \quad \ldots \quad (A)$$

Then clearly

$$
\begin{aligned}
m&=ae_{11}+be_{12}+ce_{21}+de_{22}\\
\mu&=\alpha e_{11}+\beta e_{12}+\gamma e_{21}+\delta e_{22}\\
m+\mu&=(a+\alpha)e_{11}+\ldots(d+\delta)e_{22} \qquad \ldots \quad (B)\\
m\mu&=(a\alpha+b\gamma)e_{11}+(a\beta+b\delta)e_{12}\\
&\quad+(c\alpha+d\gamma)e_{21}\\
&\quad+(c\beta+d\delta)e_{22}.
\end{aligned}
$$

The set of hyper-complex numbers $ae_{11}+\ldots+de_{22}$ in which $a..d$ range independently over a domain of rationality Ω and for which addition and multiplication are defined by (B), is called a linear associative algebra over the same domain with the four units e_{11}, e_{12}, e_{21}, e_{22} subject to the multiplication Table (A).

The above can be easily generalized, when m and μ are matrices of higher order than 2.

To Sylvester, in the 19th century is due most of the development in the theory. In his lectures on universal Algebra "he distiuguished between the diffierent degree of *vacuity* and *nullity* of a matrix, replacing by these terms, the term "indeterminate" used rather vaguely, by Cayley to denote a "null" or ' vacuous" matrix. Sylvester showed also how to derive a *chain of equations* (the *catena*) from the *Identical Equation* mentioned above and the relation of these to the Latent function of two or more matrices taken in a certain order. The terms "nullity" "vacuity" and "latent function" are of such importance in Sylvester's Theory that a brief description may be given of them.

Let ϕ be the matrix $[a]^{w}_{w}$ of order w.

If all the minors of the matrix of order $w-i+1$ vanish, but not all the minors of order $w-i$, its nullity is of the order i. If $i=1$, it is said to have "simple nullity." The term "vacuity" has reference to the number of roots, *equal to zero* of the equation

$$
\begin{vmatrix}
a_{11}-\lambda & a_{12}\ \ldots & a_{1w}\\
a_{21} & a_{22}-\lambda & a_{2w}\\
a_{w1} & \ldots\ \ldots & a_{ww}-\lambda
\end{vmatrix}=0
$$

If i roots of this equation are zero, the *vacuity* of ϕ is i; if $i=1$, ϕ is said to be "simply vacuous."

The roots of the above equation are termed by Sylvester the latent roots of ϕ and the determinant written above (when it does not vanish) is called the Latent function of ϕ and written as $|\phi-\lambda|$, λ being a scalar.

The "nullity" of ϕ cannot exceed its "vacuity" and since the latent roots of $\phi-\lambda$ are severally equal to the several latent roots of ϕ *less* λ, if g is an m-tuple latent root of ϕ the vacuity of $\phi-g$ is m.

Sylvester's Law of nullity is :—

"The nullity of the product of *two* (and, therefore, of any number of) matrices cannot be *less* than the nullity of any factor, nor *greater* than the *sum* of the nullities of the several factors which make up the Product."

When we see that the "nullity"[1] of a matrix, as defined by Sylvester, is the difference between its order and its "rank" as defined in Art. 68 of Dr. Cullis's work, Sylvester's law is equivalent to "If γ_1 and γ_2 are the ranks of two matrices of order n and R the rank of their product, then

$$R \geqq \gamma_1+\gamma_2-n \qquad \ldots \quad (i)$$

$$\left.\begin{array}{l} R\leqq\gamma_1 \\ R\leqq\gamma_2 \end{array}\right\} . \qquad \ldots \quad (ii)$$

Dr. Cullis's Theorem III (generalized in Theorem IV) of Art. 71 gives only (*ii*) above, which is only *one half* of Sylvester's famous Law which has been extended by Bucheim (Phil. Mag. Vol. 18, 1884). There is no doubt that the Identical Equation and its implications have received further developments in the skilful hands of Dr. Cullis in the subsequent volumes of this great work.

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[1] Connected with a matrix of order n, are two integers, its rank and its nullity. The sum of these two is n, so that we may dispense with either one of them and most authors use the "rank" alone. It is however convenient to use the Nullity and it is the generalization of this latter conception and not of the rank, with which we have to deal in the case of Infinite and Fredholm determinants (Maxime Bocher).

Addition and subtraction of two matrices.

Dr. Cullis's development of the Theory of matrices of which only the fundamentals are given in Chapter VI, is entirely on an *independent* plan. The *similarity* and *equality* (real and conventional) of two matrices, their *addition* and *subtraction* and the *multiplication* of a matrix by a scalar number, are defined and considered in Arts. 39-41 and fully illustrated by examples. The definition of "zero" matrices given in sub-article 4 of Art. 39 is to be carefully attended to. When we write A=0, we mean A is a matrix, *every one* of whose elements is zero,—our familiar scalar zero. With this exception, *a matrix cannot be equated to a scalar number.* And it is proved further on (page 179) that "there is no matrix which can be regarded as being always equivalent to the non-zero scalar number k for purposes of multiplication."

Product of two matrices.

In Arts. 42-46, a *product* of two matrices is defined and the chief properties of such a product are discussed. In Art. 42, *active* and *passive* rows of the factor matrices and the *activity* and *passivity* of a factor matrix are defined (cf. similar notions in Vol. IV of Sylvester's collected works, page 210). The Law of the formation of the product, $[c]_m^n$ of two matrices $[a]_m^r$ and $[b]_r^n$ of *equal passivity* r is defined by

$$c_{ij}=\Sigma_u a_{iu} b_{uj} \ldots (u=1,\ 2, \ldots r)$$

This definition is equivalent to the *two* following statements :—

(1) The *product* matrix has the *same number of horizontal rows* as the first factor matrix and the *same number* of *vertical rows* as the second factor matrix.

(2) The element c_{ij} common to the i^{th} horizontal row and the j^{th} vertical row of the product matrix, is the sum of the products obtained by multiplying each element of the i^{th} horizontal row of the first factor matrix by the corresponding element of the j^{th} vertical row of the second factor matrix.

When the passivities of the two factor matrices are *not equal*, sub article 3 of Art. 42 puts clearly how the product is to be formed by (*i*) the *insertion* of *additional* final *passive* rows in the factor which has the smaller number of passive rows or (*ii*) by striking out the *redundant* final passive rows in the factor with the larger number of passive rows, these two processes being shown to be equivalent. This leads to the generalized statement :—

$$[a]^{r}_{m}\ [b]^{n}_{s} = [c]^{n}_{m}$$

where $$c_{ij} = \Sigma_u\, a_{iu}\, b_{uj},$$

u receiving all integral values from 1 to the *smaller* of the two numbers r and s.

After giving in Arts. 43 and 44 respectively, the properties of passive and active rows in a product of two matrices (which follow easily from the definition given above), the author proves in Art. 45 two well-known propositions in the Theory of matrices *viz.*,

(*i*) The product of two matrices taken in *specified order* is distributive (sub article 2)

(*ii*) The product is in *general* not commutative (sub article 3). The Note appended to sub article 3, however, gives two instances of *special* commutative products, *viz.*,

(1) The product of a square matrix and its conjugate reciprocal (sub article 6 to Art. 46)

(2) The product of two square matrices of the same order when such product is a non-zero *scalar* matrix (example *i* of Art. 67).

On this subject of commutativity, the memoir of John Byrnie Shaw, the reputed author of a masterly "synopsis of linear Associative algebras" and of the "Philosophy of Mathematics," in Vol. 4 of the Transactions of the American Mathematical Society (pp. 251-287) on the Theory of Linear Associative Algebra may be usefully consulted.

Art. 46 deals with some *special* cases of a product of two matrices which are of frequent use in the Chapters which follow.

One noteworthy feature of the author's treatment of his subject is that he leads his reader on *step by step*, from particular cases to more general ones. After dealing exhaustively with the product of *two* matrices, he introduces the product of three or more matrices, very suitably calling that a *chain* of matrix factors, and showing how what has gone before is applicable to the chain. In particular, he proves (Art. 47) the Associative property of a product of three matrices and the next Article extends the proof to the product of any number of matrices. In these two articles, he also proves the *Distributive* property of any product.

Chain of matrix factors.

In the case of a *chain* of matrix factors, it is found necessary, for obvious reasons, to redefine, *active* and *patsive* rows, *activity* and *passivity* and the *efficiency* of the product (Art. 49).

One-to-one correspondence between the active rows of the two *extreme* factors in a chain and the active rows of the product matrix, is established. Similar correspondence between the passive rows of any two and three consecutive factor matrices in a chain is also established in Art. 49 (Cf. what goes before in the case of a product of two matrices in Art. 42, page 163). Finally a *standard product* is defined, although it is shown in the next Article (50) that the consideration of products which are not of standard form is superfluous. The remaining articles of Chapter VI deal with the properties of passive and active rows in *any* product formed by a chain, of the conjugate of a product of any number of matrices in a specified order, (Cf. the corresponding case in Quaternions which is only a special case of Matric algebra) and of special cases. It is essentially necessary that the student studying the Calculus of matrices should fully *assimilate* the contents of this Chapter, if his further progress in that Calculus is to be assured.

Chapter VII follows Chapter VI in natural sequence. We have a chain of matrix factors. The product is a matrix bearing definite relations to the chain. But a matrix is always associated with its determinoids—that function which is the *content* of a rectangular matrix. How, then, are the determinoids of the various factors related to the determinoid of the product matrix? Dr. Cullis starts, as is the practice with him, accomplished teacher that he is, with the simplest case, *viz.*, with two factors A and B and proves the following theorems:—

(1) If the *passivity* of the product AB is *less* than its *efficiency*,

$$\det AB=0$$

(2) If the passivity is *not less* than its efficiency η, then, det AB $=\Sigma(-1)^{w+w'}\Delta\Delta'$ where Δ and Δ' are the determinants of two square matrices A′, B′, to the product of which, AB can be reduced by striking out corresponding passive rows in both the factors and active rows, in the larger factor matrix and w, w', are the affects of A′, B′, in A, B, respectively.

The investigation of these two theorems is divided into several possible cases including the case when the passivity is equal to the efficiency which of course falls under (2) above. The following results are deduced :—

(*a*) If A and B are both square matrices

$$\det AB = \det A \times \det B$$

(*b*) If the *smaller* of the two matrices A and B is a square matrix or if the passivity of the product is equal to its efficiency,

$$\det AB = \det A \times \det B.$$

Three *alternative* rules are given (Theorem D, page 217) for evaluating the det AB when the passivity is not less than the efficiency and it is shown how these rules can be made applicable to the other case, *viz.*, when the passivity is less than the efficiency. These rules are patterns of Art and are fully illustrated by examples. The essence seems to be the reduction of rectangular matrices to square or inversely similar (Art. 39) matrices by permissible "striking out" of rows.

Having established the above, Dr. Cullis passes on to the treatment of the more general case of the determination of the determinoid of the product of *any number* of given matrices in Art. 58. He gives six theorems, covering all possible cases but not being content with the same, gives in the next article, a practical plan for the *progressive* development of the determinoid.

Art. 60 deals with some special cases, in which one of the factors is a unit matrix, a scalar matrix or has a numerical factor. And in Art. 61 taking, without loss of generality the product of four factor matrices, A, B, C, D whose two activities are each equal to the efficiency (η) of the product, proves that

$$\det A\,B\,C\,D = \det A'\,B'\,C'\,D'$$

where A′, B′, C′, D′ are *matrices whose elements are the derived determinants of order η which can be formed from the matrices A, B, C, D respectively and where each of the extreme factors A', D', contains only one active row.*

For a proper understanding of this very important theorem which is really a concise way of expressing the rather complicated results of Theorem VI of Art. 58 (pp. 225-229), the terms "vertical and horizontal minor matrices" are introduced in pages 233-234 and a scheme is laid down for the choice of successive minor matrices of the factors for forming the equation above.

The Theorem, the new terms and the scheme are illustrated by the working out of some examples.

The Chapter concludes with an investigation of "the algebraic sum of the affected minor determinants of order k of a product of given matrices." This is an important investigation for subsequent work and the two theorems which are given are clearly shown to be *generalizations* of those obtained in Arts. 56 and 58.

We can derive, from a fundamental matrix $[x]_m^n$ minor determinants of a given order which is less than either m or n. With these unaffected minor determinants we can construct matrices but definite schemes of formation are required. In Art. 65(1), (Chapter VII), the definition of a *complete* matrix of the minor determinants is given. According to this definition the complete matrix is $[\epsilon]_\mu^\nu$

where

$\epsilon_{pq} = (x_{mn})_k^k$ where the sequences $[m_1\ m_2\ \ldots\ m_k]$ and $[n_1\ n_2\ \ldots n_k]$ have their pth and qth values

$$\mu = \binom{m}{k}$$

$$\nu = \binom{n}{k}$$

Therefore, the complete matrix can be formed in $\mu!\ \nu!\ k!\ k!$ ways; schemes of formation of complete matrices are required. This is supplied in sub article 2. Sub Art. 3 gives what are called standard schemes of formation and standard matrices of minor determinants. In sub article 4 complete matrices of affected minor determinants of given order are dealt with showing the *complete* character of the work under review. These definitions and formation of schemes are introductory to the two remarkable theorems which are given in Art. 66, of which, what has been given in formula (D) of Art. 61, Chapter VII, is only a particular case. The theorems are given below :—

$$\text{Let } X = A\ B\ C\ ..S\ T$$

be a standard product in which the passivities are not subject to any restrictions. Let η be the efficiency of the product X and let K be any number which is not greater than η so that the product has derived determinants of order K.

Theorem I. If any one of the passivities is less than K, then every *complete* matrix of the minor determinants (unaffected or affected) of order K, of the product matrix X is zero.

Theorem II. If *no one* of the passivities is less than K, then every complete matrix of the minor determinants (unaffected or affected) of order K of the product matrix X, is equal to the product of any set of *correspondingly* formed complete matrices of the minor determinants (unaffected or affected) of order K of the factor matrices A, B, C ...... S, T. The nature of the correspondences alluded to in Theorem II is indicated in the proof.

Although in the enunciation the product is taken to be a standard one, it is shown that the theorems remain true when the product is not of the standard form (note 3, page 258).

Chapter VIII concludes with two theorems on the reciprocal and conjugate reciprocal of a standard product of *square* matrices. These theorems are :—

(1) The reciprocal, of a standard product, of any number of square matrices taken in a given order is equal to the product of the reciprocal of the factor matrices taken in the *same* order.

(2) The conjugate reciprocal, however, is equal to the product of the conjugate reciprocals taken in the *reverse* order. (*Cf.* Art. 54, sub Art. 2, Chapter VI).

The proofs of these theorems depend on the general theorem of Art. 66 but an alternative proof also is given. Some instructive examples of commutative products are also given.

Rank and Connection.

Chapter IX is very important in the Calculus of matrices; and the student will do well to ponder every definition that is given and every theorem that is proved. It treats of the "Rank of a matrix and the *connections* between the *rows* of a matrix." After the student has carefully worked through the Chapter he cannot but feel grateful to Dr. Cullis for the satisfactory manner in which he has dealt with his subject.

"If A = $[a]_m^n$ be a matrix and s be any number which is not greater than the efficiency of the matrix, and, if *all* the derived determinants of order s of the matrix vanish, then all derived determinants of orders greater than s vanish; for, every such determinant can be expanded in terms of derived determinants of order, s."

"The rank of the matrix A is the greatest order which a *non-vanishing* derived determinant can have. Thus, the rank of the matrix A is γ when A has at least one derived determinant of order γ which does not vanish, whilst every derived determinant of order $\gamma+1$ vanishes and therefore every derived determinant of order greater than γ vanishes."

Obviously the rank γ of a matrix cannot exceed its efficiency, η when $\gamma < \eta$, the matrix is called "degenerate" when $\gamma=\eta$, it is called "undegenerate." An undegenerate has non-vanishing derived determinants of all orders not exceeding η but its determinoid may or may not be zero. In a degenerate of efficiency η, all derived determinants of order η vanish; also the determinoid of the degenerate vanishes; for the determinoid can be expressed in terms of the derived determinants (Art. 30, Chapter V) of order η.

"A matrix whose determinoid vanishes is called a "singular" matrix. A matrix whose determinoid does not vanish is called "non-singular." A degenerate matrix is necessarily singular and a non-singular matrix is necessarily undergenerate."

A determinoid is said to have the same rank as its matrix (Art. 68).

These preliminaries are rightly followed by twelve illustrative examples.

Art. 69 is important in as much as it establishes "*connections* between the rows of a matrix whose elements are constants," and thereby links Dr. Cullis's really original method of the treatment of matrices with the genesis of the Theory in Cayley's Memoir of 1858.

Consider the matrix $[a]_m^n$: There is said to be a *connection* between (or among) the *horizontal rows* or their matrices when there exist scalar multipliers, h_1, h_2, … h_m, not all zero, such that

$$h_1\,[a_{11}\,a_{12}\,..\,a_{1n}] + \dots h_m\,[a_{m1}\,a_{m2}\,\dots\,a_{mn}] = \text{matrix zero.}$$

This matrix equation is equivalent to the m scalar equations:—

$$\left.\begin{array}{l} h_1\,a_{11} + h_2\,a_{21} + \dots h_m\,a_{m1} = 0 \\ h_1\,a_{12} + h_2\,a_{22} + \dots h_m\,a_{m2} = 0 \\ \dots\qquad \dots\qquad \dots\qquad \dots \\ h_1\,a_{1m} + h_2\,a_{2m} + \dots h_m\,a_{mm} = 0 \end{array}\right\}(A)$$

which set can obviously be written, in Dr. Cullis's notation, as

$$[h_1\,h_2\,\dots\,h_m]\,[a]_m^n = 0.$$

Similarly there is said to be a *connection* between the *vertical* rows, or their matrices, when there exist scalars k_1 … k_n not all zero, such that

$$k_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \dots k_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} =$$

matrix zero.

This is equivalent to m scalar equations among the n variables of the type

$$\Sigma\,k_r\,a_{ur}\,\dots\begin{pmatrix} u = 1, 2 \dots m \\ r = 1, 2 \dots n \end{pmatrix}$$

which can be written in the Matrix form $[a]^n_m \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = 0$

If n is $> m$ there will be, in the case of connection between horizontal rows $n - m$ more scalar equations of the given type but these are not dealt with, for it is not the object of the learned writer to discuss, at the stage, the theory of Linear dependence; on the other hand his object is obviously to erect an *independent* Theory of Matrices on a *logical* basis and, then, to apply the same to the discussion of Linear dependence and linear algebraic equations which he does in the last Chapter of the First Volume.

Art. 70 gives eight theorems concerning *connections*. These are important for the subsequent handling of matrices. Theorem I states that if there is any connection between the horizontal or vertical rows of a *square* matrix, the determinant of the matrix vanishes. This is fundamental and the proof is given, not by the Theory of Linear algebraic equations as is the usual procedure but by the author's Theory of matrices. Theorem II, *viz*., that if Δ is any *non-vanishing* derived determinant of a matrix A, then the horizontal rows and also the vertical rows of A which occur in Δ are *unconnected*. This might have been a corollary to Theorem I. Theorem III is practically a generalization of Theorem II. As this theorem is important we may give its enunciation :—

"If γ is the *rank* of a matrix A and if Δ_γ is one of its non-vanishing derived determinants of order γ then;

(*i*) the horizontal (and also the vertical rows) of *A* which occur in Δ_γ are *unconnected*;

(*ii*) all other horizontal (and similarly all other vertical) rows of A are connected with those occurring in Δ_γ.

The other theorems, simple and important as showing the dependence of the *rank* of a matrix on the *connections* between rows, are easy deductions, the last Theorem, *viz*., that "if a *connection* exists between any of the *active* rows of *either extreme* actor matrix in a product formed by a chain of matrix factors, then there is exactly the same *connection* between the corresponding rows of the product matrix," is a *generalization* of the properties of active rows considered in sub arts. 9 and 10 of Art. 53. It is noted that the *converse* of this Theorem is not true.

Art. 71 introduces seven theorems most of which serve to facilitate the *determination of the rank of a given matrix*; they are of fundamental importance to a student of the Calculus. In example XII, page 285, is introduced the notion of *equipotent* matrices. "Two similar matrices are called *equipotent* when either, and therefore, each, can be derived from the other by prefixing and post-fixing undegenerate square matrices. The necessary and sufficient condition is clearly set down. Equipotent matrices have equal ranks.

In note I on page 288, are given the *necessary* and *sufficient* conditions that a given matrix, A, shall have rank, γ, *viz.* :—

(1) That A shall have a non-vanishing derived determinant Δ_γ of order γ.

(2) That *every* derived determinant of order $\gamma + 1$ of A which contains Δ_γ as a minor, shall vanish.

Although these conditions follow from Theorem I of Art. 71, a more direct proof is said to be given in "Ex. V of Art. 110" and in Note 2, page 288, it is said that special theorems relating to the rank of symmetrical or self-conjugate square matrix are given in Art. 119. The First volume contains only 97 articles and these references probably relate to the second volume but the reviewer does not find that they are correct. This however has been explained on page vii of the Preface to the second volume.

Extravagant matrix.

Art. 72 deals with the rank of a product of two mutually conjugate matrices and in clause 2 of the same, is introduced the expression "*Extravagant Matrix*." As such matrices play a considerable role, we note that "an undegenerate matrix, in which the sum of the squares of the simple minor determinants is equal to zero, is an extravagant matrix." Clearly no matrix whose elements are all real or all pure imaginaries, can be extravagant.

Art. 73 gives two theorems showing how the rank of a complete matrix of the minor determinants of a given order of a matrix can be determined, when the rank of the latter is known.

The chapter closes with some elementary investigations (Art. 74) on the rank of a matrix and connections between its rows when its elements (hitherto treated as constants) *are rational integral functions of certain variables.* The importance of this article for subsequent work cannot be over-estimated.

Chapter X deals with matrix equations of the First degree. The most general form of such an equation is

$$\Sigma_1^r A_\gamma X B_\gamma = C, \text{ where } A_1, A_2, \dots A_\gamma, B_1, B_2, \dots B_\gamma$$

and C are known matrices whose elements are constants. This general equation is not discussed in this Chapter but the form that is dealt with is A X B = C of which *particular* cases are A X = C and X B = C. These three receive very full treatment. In Arts. 75-77, definitions are given of a *solution,* the *general* solution, a *finite* and an *infinite* solution and equations of the specially simple form X + A = B are considered. The special cases of A X = C, X B = C and A X B = C in which A and B are *undegenerate* square matrices are first considered in Arts. 78-80 and, then, the general cases, when A and B are any matrices whatever, are discussed in the three following articles. To a student who has studied Quaternions the most general form $\Sigma_1^r A_\gamma X B_\gamma = C$ of the matrix Linear will recall, the Quaternion equation $\Sigma A q B = C$ the operator, $\phi = \Sigma A (\ ) B$ called by Sylvester, the Nivellator, and its quadrate Matrix and how Cayley has shown that the "Theory of the solution of the Linear equation in *quaternions* enters into relation with that of the solution of the Linear equation in Matrices." (Vide Cayley's contribution on the Analytical Theory of Quaternions in the Third Edition of Tait's Quaternions, pp. 148-152).

Law of cancellation of matrix factors.

The chapter closes with a summary of cases in which it is allowable to *cancel* a matrix factor which is *common* to both sides of a matrix equation. Art. 84 gives very *precise statements* (six in number) which should be carefully followed and borne in mind, regarding the *Law of Cancellation* of matrix factors in a matrix equation. We notice in the enunciation of the Law the importance of the notions of *rank* and *passivity*. Art. 85 follows, in natural course, with similar statements regarding the cancellations of matrix factors in a matrix *Identity* which too receives a definition. A thorough comprehension of the Law of Cancellation is essential to progress in the General Theory of rectangular matrices.

Solutions of any system of Linear algebraic equations by the help of the Theory of matrices.

Chapter XI, the last in the First Volume, treats of the solution of *any system* of Linear algebraic equation. This is a really valuable Chapter and the reviewer has not seen this important subject dealt with in the same masterly and refreshing manner in any of the modern English works in Algebra. The late Maxime Bocher's Introduction to Higher Algebra, the best of its kind in recent years, gives only the *beginnings* of the general theory. Starting with *connections* (clearly defined) between linear *functions* and between linear *equations* (Art. 86), Dr. Cullis gives, as a first treatment, the solution of any system of linear equations and writes the *single* matrix equation which is equivalent to the *m* scalar equations of the type:—

$$a_{11} x_1 + a_{12} x_2 + \dots a_{1n} x_n = c_1.$$

. Since there is one—one correspondence between the solutions of the system of scalar equations and the solutions of the matrix equation, the solutions of the scalar equations depend on the solutions of a matrix equation of the first degree of the form A X = C, discussed in the previous Chapter. Stating clearly what are the *unaugmented* and *augmented* matrices of the system of the scalar equations, the theorems arrived at, in regard to the matrix equation, in Art. 81 of Chapter X, are translated to apply to the scalar equations and six note-worthy results are given including the *infinite* solutions. The author next proceeds to find the *general* solution when finite solutions exist and discusses two cases, the principles being fully illustrated. A second treatment of the solution of a system of Linear equations of the type:—

$$a_{11} x_1 + a_{12} x_2 + \dots a_{1n} x_n + a_{1\,n+1} = 0$$

is given in Art. 88 and similarly dealt with. The next article discusses the solution of any system of *homogeneous* linear equations which are of such frequent occurrence in analysis. In Arts. 90-93 *complete sets* of unconnected solutions of any system of linear algebraic equations are defined and by means of them a symmetrical form is given to the general solution. The various examples worked out in connection with Art. 90, introduce the ideas of 'extravagant' and "non-extravagant" equations and solutions, although the reviewer would have wished an "Extravagant solution" to be *formally* defined as an "Extravagant equation" is in Ex. vii, page 389. In Arts. 94 and 95 properties of mutually *orthogonal* solutions are considered and it is shown, in particular, that a system of linear equations, whose matrix is *real*[1] (*i.e.* a matrix all of whose elements are real) has always a *complete set* of unconnected mutually orthogonal real solutions. Connections between linear equations receive *still another interpretation* in Art. 96 and the last Chapter, which is probably the only chapter in the book which contains *applications* of the Theory discussed in the previous Chapters, closes with a discussion of the conditions for the existence of *functional* dependences between algebraic functions.

The Second Volume.

The Second Volume has been published in 1918. "The continuation of the work" we are told in the interesting preface "had been greatly hindered by untoward circumstances, above all by the difficulty of obtaining sufficient leisure for the final preparation of the manuscript for the Press." The theoretical portion has been constantly increased, in the first place by *abstractions* from the applications and, in the second place by incorporating the work of other writers. The learned author decided, with reluctance (for, his original aim was "to give a systematic account of certain *applications* of Matrices to illustrate the very great advantages gained by using them in almost all branches of Mathematics"), to publish, as a Second Volume, the first half of that portion of the complete work which deals in greater detail with the Theory of Matrices. The Second Volume, accordingly, contains those parts of the Theory which naturally precede any investigation of the special properties of functional Matrices (Matrices whose elements are rational integral functions of a finite number of variables). It deals almost exclusively with Matrices whose elements are constants, which may be arbitrary parameters, and with those transformation of such Matrices which are classed as equigradent.

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[1] The idea of a domain of rationality Ω should be fully grasped, in order to understand what a *general* theorem concerning matrices becomes in the special case when the matrices are *real*. When the operations on the elements are only *rational* the general theorem can be made at once applicable to the real matrix by taking Ω to be the domain of all real numbers. But it should be noted that the real domain has properties which are not possessed by other domains and special treatment is necessary. (This subject has been further developed by Dr. Cullis in Chapters XVI and XVII of his second volume.)

The Volume contains eight Chapters, and like the first, has a full and useful index, and a summary of the contents of each Chapter prefixed to it. There are three appendices,—Appendix A, on the utility of the relations obtained in Chapter XIII, *viz.*, between the elements and minor determinants of a matrix; Appendix B, on the Pfaffian of a Skew-Symmetric matrix of even order; which was suggested to Dr. Cullls by his perusal of a paper by Mr. Haripada Datta (on Symmetric determinants and Pfaffians, Research Paper No. 5 in the Mathematical Department of the University of Edinburgh) and Appendix C,—on equigradent transformations in which one of the transforming factors is a semi-unit matrix.

Dr. Cullis has usefully anticipated the reviewer, by giving, on pp. viii-x, remarks concerning the contents of the *individual* Chapters. In the Preface to the Second Volume, he has taken the student of Higher algebra more into his confidence and admitted him into his work-shop so that he may take a glance at the materials with which he is raising his edifice—not finished yet but of magnificent proportions and bidding fair to end in an outstanding land-mark in the Mathematical literature of the present century!

We note that although "the present volume is avowedly restricted to the *Theory* of matrices, it actually presents a large number of Geometrical *Applications* and of algebraical *Applications* to which, it is promised, attention will be subsequently directed." Thus, "any set of matrices which are *vertically equivalent* to one another are regarded as defining a "Spacelet,"[1] which is completely represented by *any one* of them, usually by one which is undegenerate, the "Spacelet" being a "point" when the common rank of the matrices is 1. And any property of a matrix which remains unaltered when the matrix is replaced by any matrix vertically equivalent to it, is a property of a "Spacelet" and conversely all properties of "Spacelets" are properties of "matrices." "Spacelets" and "points" are spoken of, from the outset, as *geometrical* concepts although their *definitions* are not so. The object expressly is to "lighten subsequent Chapters on the geometrical applications of matrices" such as to *n*-dimensional geometry. In Chapter XVI which is on '*equigradent* transformations,' it is shown that "every such transformation of a matrix whose elements are constants, corresponds to a *linear* transformation of the variables in a *bilinear* or *quadratic* algebraic form (Note 6, page 234). This is a note-worthy application of the Calculus of matrices, on the algebraic side.

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[1] In generalized Vector analysis, a matrix of n short rows and m long rows may be considered as given by m Vectors of an n-dimensional space and there is no doubt many of the theorems arrived at may be obtained expeditiously from the point of view of such analysis but the author's stand-point is quite *different* and this fact must not be over-looked by a reviewer.

On the geometrical side, the notion of "Spacelets" plays a great part in the Second Volume. The 'rank' of a 'spacelet' the 'paratomy' of two spacelets and the relations between 'spacelets' represented by the terms '*intersection*' and '*join*' and by the terms '*incident*' and '*connected*,' are properties which depend only on the notion of *connection* and are *invariant* in every equigradent (or projective) transformations of the points of space. The "extravagance" of a 'spacelet,' the 'orthotomy' and 'cross-rank' of two 'spacelets' and the relations between 'spacelets' represented by the terms 'core' and 'plenum,' 'orthogonal' and 'normal,' are properties which depend on the notion of '*orthogonality*' and are invariant in every '*semi-unit transformation*' of the points of space, *i.e.*, equigradent transformation which leaves the Absolute Quadric unaltered. These are indeed valuable contributions to the much-looked for *applications* of the Calculus of matrices and as, aids to the imagination, which is often baffled in dealing with hyper-space problems, 'spacelets' and their intersections are graphically represented by areas, a *shaded* area always representing a *completely extravagant* spacelet, *i.e.*, a spacelet which being *orthogonal* with itself, is a generator of the Absolute Quadric.

Compound and Compartite matrices.

Notation.

The opening Chapter (XII) which deals with 'compound' and 'compartite' matrices, gives notations for such matrices and their determinoids which are the complete generalizations of the notations used in the First Volume. A convenient and expressive notation is of the first importance, alike to the mathematician who propounds a Theory and to the student, who tries to understand him. Liebniz recognised early the value of form.

Notations and algorithms held a prominent place in his conceptions and he wrote in 1676 that "one part of the secret of analysis consists in the *characteristic, i.e.*, to say in the Art of employing well the symbols we use. As Laplace said "the *language* of analysis, most perfect of all, being a powerful instrument of discoveries, its notations, specially when they are necessary and happily conceived, are so many germs of new calculi." In the development of a subject like the matrix theory, sufficient stress cannot be laid on the importance of a well-chosen notation. "It rests with the worker in mathematics not only to explore new truths but to devise the language by which they may be discovered and expressed; and the genius of a great mathematician displays itself no less in the notation he invents for deciphering his subject than in the results attained." And one cannot but have faith in "the power of well-chosen notation to simplify complicated theories and to bring remote ones near." The notations for matrices and determinoids which Dr. Cullis has elaborated in his work, fulfil the requirements of a useful instrument of research and are certainly an improvement on the notations of Cayley and of other previous writers.

Relations between the elements and the minors

Spacelets.

Full of interesting matter is Chapter XIII, dealing with the relations between the elements and minor determinants of a matrix. Starting with the determination of the connections between the short rows of an undegenerate matrix (Art. 108), we are led to a host of useful relations which are finally seen to be deducible from Art. 116 headed "Identical relations between the elements of any matrix." In Arts. 110 and 111 are given *Sylvester's identities* (a reference to the Collected works of Sylvester where these occur had been useful) involving the "primary superdeterminants" (vide note 1 on page 17, Vol. II) of a determinant and the corresponding identities involving the "primary sub-determinants" (also defined in Note I on page 17) of a determinant. Conditions for the *equivalence* of two similar undegenerate matrices and the *equivalence* of two systems of linear algebraic equations are given in Arts. 113 and 114, the sign '$\equiv$' being introduced to denote the equivalence of two matrices. Note 3 to Art. 113, pages 78-80 is specially interesting as introducing the notion of undegenerate matrices as "spacelets." "An undegenerate matrix $\overline{\underline{\,a\,}}{}^{\gamma}_{n}$ of rank γ where γ is $< n$ represents a flat locus of $\gamma - 1$ dimensions in homogeneous space of $n-1$ dimensions, *i.e.*, it represents a certain $(\gamma-1)$-way subspace or 'spacelet' w_γ of the complete $(n-1)$-way homogeneous space w_n. γ is called the rank of the 'spacelet' w_γ and, in particular n is the rank of the complete space w_n The elements of the successive γ vertical rows of $\underset{\llcorner\lrcorner n}{\overset{\ulcorner\urcorner\gamma}{a}}$ are the projective coordinates of γ unconnected points of w_n which lie in w_γ and competely determine w_γ. Every other similar undegenerate matrix $\underset{\llcorner\lrcorner n}{\overset{\ulcorner\urcorner\gamma}{b}}$ which is equivalent to $\underset{\llcorner\lrcorner n}{\overset{\ulcorner\urcorner\gamma}{a}}$, *i.e.* when $\underset{\llcorner\lrcorner n}{\overset{\ulcorner\urcorner\gamma}{b}} = \underset{\llcorner\lrcorner n}{\overset{\ulcorner\urcorner\gamma}{a}} \underset{\llcorner\lrcorner \gamma}{\overset{\ulcorner\urcorner\gamma}{k}}$ where $\underset{\llcorner\lrcorner \gamma}{\overset{\ulcorner\urcorner\gamma}{k}}$ is undegenerate, represents the same spacelet w_γ. Examples IV and V of Art. 113 are of 'spacelets' in homogeneous 3 and 4 dimensional spaces.

Square matrices.

Chapter XIV gives an account of some special properties of a product of *square* matrices. The Chapter opens (Art. 118) with the recapitulation of the properties of a product of square matrices already proved in Vol. I. Art. 119 defines two *cojoint complete* matrices of the minor determinants of a square matrix. In this and some of the following articles are given a number of their properties. Art. 124 gives a separate account of the properties of the reciprocal of a square matrix and the next three articles deal with properties of symmetric and Skew-symmetric matrices (Appendix B should also be read in connection with these) criteria for the determination of their ranks being given. Anent Skew-symmetric matrices the reviewer would like to invite attention to the following prize problem announced in 1919 by the Philosophical faculty of the University of Berlin :—

Skew-Symmetric matrices.

A Prize-problem.

"To determine by means of the *Theory of Elementary divisors*, the criteria that a given matrix be capable of representation as the composition (product) of two Skew-symmetric matrices."

(*N.B.*—Prof. A. A. Bennett discussed the problem in the Bulletin of the American Mathematical Society for July 1919, pp. 455-458. He has acknowledged, in March 1920 number of the Bulletin, the reviewer is glad to notice, Harold Hilton's prior solution in Section 8, Chapter VI of his well-known work on "Homogeneous Linear substitutions").

Chapter XV deals with the possible *ranks* of the product matrix and the factor matrices, in any matrix product. "The theorem of Art. 133 and the final results of Articles 135 and 137 constitute the *complete generalization of the Law of cancellation for matrices*." The concluding articles deal with the *equivalences* of matrices on which the definition of a 'spacelet' is based and with the "joins," "intersections" and "connections" of matrices and spacelets. These require a careful study as introductory to the geometrical applications of the Calculus of matrices.

Law of Cancellation of matrices—its generalization.

Chapter XVI is on equigradent transformations of matrices and the use thereof in the reduction of a matrix of rank γ to standard forms. The transformation of $[a]^s_r$ into $[b]^n_m$ by means of an equation of the form $[b]^n_m = [h]^r_m \, [a]^s_r \, [k]^n_s$ in which $[h]^r_m$ and $[k]^n_s$ are undegenerate matrices with constant elements having ranks r and s equal to their respective passivities, is called an *equigradent* transformation. This definition not only holds when the matrices have constants as their elements but also when they have *rational integral functions of certain variables as their* elements. In the latter case it can be shown that an equigradent transformation leaves the minimum degrees of connection of the matrix transformed unaltered. Hence the name 'equigradent.' Two equigradent matrices have necessarily the same rank.

Equigradent transformations.

The simplest standard form is a similar matrix which is conventionally equal to the unity matrix $[1]^r_r$ and it is shown that every matrix of rank r with contant elements, can be obtained from the unit matrix, by an equigradent transformation, which reminds one of the development of the Theory of integers from unity. The reductions of symmetric and Skew-symmetric matrices by symmetric equigradent transformation receive careful attention. This is an important and difficult Chapter and the student will do well to grasp thoroughly the part played by the notion of Domain of rationality Ω, in the transformations.

Matrix equations of the Second degree.

Chapter XVII is on the solution of matrix equations of the second degree. "One of the most important results obtained in this Chapter, is the *general* formula for all solutions of any assigned rank ρ of the symmetric equation $\overline{\underline{x}}{}^{s}_{m}\,[x]^{m}_{s} = \overline{\underline{a}}{}^{r}_{m}\,[a]^{r}_{m}$ where $[a]^{m}_{r}$ is a given matrix of rank r (Art. 157). This is important, in as much as, the general Theory of *extravagant* matrices is chiefly based on this result and leads at once to the reductions in Chapter XVIII.

The reviewer cannot pass without notice the important notions introduced in Art. 155 of a "Semi-unit matrix" and of a "square semi-unit matrix." The examples attached to this Article are specially interesting as showing the power and elegance of the Calculus when applied to such familiar things as rotations of a rigid body in 3-way space.

Extravagance and degeneracy.

Chapter XVIII, the last but one Chapter in the Second Volume, deals with the very important notions of "the degeneracy" and "the extravagance" of a matrix. The "degeneracy" is the amount by which the rank of a matrix falls short of its "efficiency." In sub-article 2 of Art. 72 in the First Volume, an "extravagant matrix" had been defined but it is necessary to define what is the "extravagance" *of* a matrix. It is the degeneracy of the product $\psi = [a]^{n}_{r}\,\overline{\underline{a}}{}^{r}_{n}$. The "extravagance" of a spacelet (or the degree of its orthogonality with itself) defined in Art. 170 is that property of it which is next in importance to its "rank." It is invariant in every semi-unit transformation of the points of space and can be interpreted being the "rank" of contact of the 'spacelet' with the Absolute Quadric. The Chapter deals with interesting geometrical notions, the "core" and "plenum" of a 'spacelet' being defined. With every 'spacelet' is associated a completely *extravagant* "spacelet" called its 'core' which is the locus of all points which lie in the given spacelet and are orthogonal with it, *i.e.*, the locus of the points in which the given spacelet touches the Absolute Quadric and a *plenarily extravagant* (vide note 2, page 382) spacelet called its "*plenum*" which is the smallest spacelet containing the given spacelet and all points orthogonal with it. Mutually "orthogonal" spacelets are distinguished from mutually "normal" spacelets. A given spacelet has *one* and only *one* normal while an *indefinite number* of spacelets are orthogonal with it.

On the algebraic side the reduction of a matrix whose elements are constants to a standard form by a unilaterally semi-unit equigradent transformation is specially valuable, as it enables us to complete the discussion of the unconnected mutually orthogonal solutions of any system of homogeneous linear algebraic equations which had been left unfinished in the closing Chapter of the First Volume.

Paratomy and orthotomy.

Chapter XIX is on the "Paratomy" and "orthotomy" of two matrices and of two spacelets of homogeneous space. Two matrices $\underset{\llcorner\!\lrcorner\, n}{\overset{\ulcorner\!\urcorner\, p}{a}}$ and $\underset{\llcorner\!\lrcorner\, n}{\overset{\ulcorner\!\urcorner\, q}{b}}$ of ranks r and s which have the same number of vertical rows or two matrices $[a]^{n}_{p}$ and $[b]^{n}_{q}$ of ranks r and s which both have the same number of horizontal rows, are to be regarded as having the same mutual 'paratomy' and the same mutual 'orthotomy' as the homogeneous spacelets $w_r \equiv \underset{\llcorner\!\lrcorner\, n}{\overset{\ulcorner\!\urcorner\, p}{a}}$ and $w_s \equiv \underset{\llcorner\!\lrcorner\, n}{\overset{\ulcorner\!\urcorner\, q}{b}}$. The author, therefore, confines himself to a consideration of the paratomy and orthotomy of spacelets in homogeneous space. The mutual paratomy, p_1 of any two spacelets w_r and w_s in space w_n is defined to be the greatest number of unconnected points common to w_r and w_s. It is the rank of that spacelet which is the locus of all points common to w_r and w_s, *i.e.*, the rank of the complete intersection of w_r and w_s. The *mutual orthotomy* of any two spacelets $w_r \equiv \underset{\llcorner\!\lrcorner\, n}{\overset{\ulcorner\!\urcorner\, r}{a}}$ and $w_s \equiv \underset{\llcorner\!\lrcorner\, n}{\overset{\ulcorner\!\urcorner\, s}{b}}$ of space w_n is defined to be the *degeneracy* of the product $\psi = [a]^{n}_{r} \underset{\llcorner\!\lrcorner\, n}{\overset{\ulcorner\!\urcorner\, s}{b}}$ and the rank of ψ is called the *cross rank* of spacelet w_r and w_s. The most noteworthy results are those relating to the *greatest* possible mutual 'orthotomy' of two spacelets and the *independence* of the extravagances of two spacelets of given ranks which have a given complete intersection. "Real spacelets" are dealt with in Articles 181 and 182 (notes appended thereto) and it is easy to see that all the abstract theorems of the last Chapter are applicable to the common metrical space Ω_{n+1} of n dimensions when we define the paratomy and orthotomy of two spacelets of this space, to be those of their infinite sub-spaces, *i.e.*, those of their intersections with the infinite sub-space w_n of the space of n dimensions. The Chapter closes (Art. 183) with a treatment of the properties of mutually orthogonal spacelets.

Applications of the Theory of matrices.

As Sylvester said "the theory of determinants is an algebra upon algebra, a Calculus which enables us to construe and foretell the results of algebraical operations in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic." "Properly regarded, the theory of determinants" of which so much could be said, is as remarked by M. Bocher, "hardly more than a shadow of the theory of matrices and just so far as one attempts to ignore this fact does the subject become an artificial one or at best a tool for other investigations." Even as a tool, its potency is great; for, we find the theory largely entering into modern works on Algebra, Geometry and Applied Mathematics. The reviewer may be pardoned if he would mention below some few of the applications of the theory which the learned author has developed with such consummate skill :—

(*i*) The reviewer has come across a notice of a remarkable book on Analytical Geometry entitled "Cinq Etudes de Geometrie analytique" by M. Stuyvaert. Gand. E. Von Goethem, 1908, vi & 230 pp. This work consisting of applications of the Theory of matrices and elimination was awarded the "Prix François Deruyts" by the Royal Academy of Belgium. It exhibits the *singular power* and *elegance* of the Matrix theory and has received an interesting review, in pp. 479-484 of the Bulletin of the American Mathematical Society, Vol. 17 (1910-11).

(*ii*) The late M. Bocher's Introduction to Higher Algebra gives some applications of the Theory. The last three chapters of the the book, constitute an excellent study of matrices whose elements are polynomials in a single variable λ called λ matrices. The theory is applied (*a*) to the elucidation of the Theory of Elementary divisors invented by Sylvester, H. J. S. Smith and more particularly by Weirstrass and perfected by Kronecker, Frobenius and others, (*b*) to the equivalence and classification of bilinear forms and collineations and (*c*) to the equivalence and classification of pairs of Quadratic forms.

(*iii*) Prof. H. J. S. Smith's Memoir on the "arithmetical invariants of a rectangular matrix" (*vide* his collected works) is one of the earliest known applications to the Theory of numbers.

(*iv*) Application of the Theory to Linear Differential equations by J. Brill (*vide* Cambridge Philosophical Society's Proceedings, Vol. 8). This memoir is written on the basis of Cayley's developments.

(*v*) O. Veblen and Young's Projective Geometry makes much use of the Matrix Theory (*vide* Art. 54, page 156, Art. 95, pages 268 and 269 of Vol. I, and Arts. 124 *et seq* of Vol. II). There are, in this work, exceedingly interesting applications of the elements of the Theory of matrices.

(*vi*) Miller and Blichfieldts' "Finite groups," Art. 6, pp. 13-14 discusses the bearing of the theory of matrices on the Theory of groups, in an elementary way.

Blichfieldts Finite Collineation groups, Chapter I gives applications of the theory *to Liner groups.*

(*vii*) The elegant article by Dr. Cullis on "a special square matrix of order six" in the Bulletin of the Calcutta Mathematical Society, Vol. X, No. 3, gives by means of the theory, the stress-strain relations of an isotropic body without assuming the existence of a strain-energy function.

The reviewer feels that he has only imperfectly sketched the contents of the first two volumes of a remarkable contribution to the ever-growing mathematical literature of the twentieth century—remarkable alike for its breadth and depth, for its logical structure and its didactic value. The labour involved by and the earnestness of purpose brought to bear on, the making of the work have been worthy of the author whose reputation as an accomplished Mathematician and as an inspiring teacher has reflected glory on the University of Calcutta, where he holds now the exalted position of the Hardinge Professor of Mathematics. This appointment, as he tells us in the Preface to to the Second Volume, had afforded him additional leisure for the completion of the monumental work he has undertaken and it is a pleasure to note the author's expression of special gratitude to the Hon'ble Sir Asutosh Mookerjee, Kt., C.S.I., President of the Post-Graduate Councils in Arts and Science for his stimulating interest and encouragement—interest and encouragement in a really promising field of work.

But the very character of the work in its elaborateness and its novel nomenclature renders it necessary that a manual for students attacking the subject for the first time should be written. Such a manual may well be on the lines of Joly's Manual of Quaternions, in the first 20 pages of which is given the 'grammar' of the Calculus Gauss writing to Bessel in 1810 lamented " I am giving this winter two courses of lectures to *three* students, of which one is only moderately prepared, the other less than moderately and the third lacks both preparation and ability. Such are the onera of a mathematical profession. " A handy manual interspersed with obvious applications is likely to remove the possibility of such a lament.

Elaborate character of the work.

A. C. Bose.

Note on Spherical Waves of Finite Amplitude

By

Sudhansukumar Banerji

[*Read November 16th, 1919*]

The equations governing the mode of propagation of aereal plane waves of finite amplitude were first obtained by Poisson [1] as long ago as 1808, and formed the subject of further investigation by Stokes [2] in 1848. In the case of the adiabatic propagation the exact equation was first derived by Earnshaw [3] in the form

$$\left(\frac{\partial y}{\partial x}\right)^{\gamma+1}\left(\frac{\partial^2 y}{\partial t^2}\right)=a^2\,\frac{\partial^2 y}{\partial x^2}.$$

The mode of propagation of waves according to this equation has been discussed by Riemann,[1] Rankine [5] and Hugoniot.[6] In an important memoir. Rayleigh [7] gave a critical and historical account of the subject and discussed in further detail special solutions of the problems of finite waves and the influence of viscosity and thermal conductivity on their mode of propagation. It will thus appear that although the aerial plane waves of finite amplitude have received a fairly adequate treatment, the propagation of spherical waves of finite amplitude has not hitherto received any attention inspite of the fact that the differential equations for both types of disturbances bear a close resemblance to each other. In the following pages the principal features of the latter type of disturbance have been obtained.

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[1] Poisson "Mémoire sur la Théorie du Son," *Journ. de l'Ecole Polytechnique*, Vol. VII, p. 319, *et. seq.*, 1808.

[2] Stokes, "On a difficulty in the theory of sound," *Phil Mag.*, 1848. *Mathematical and Physical Papers*, Vol. II, p. 51.

[3] Earnshaw, *Roy. Soc. Proc.* 1859 ; *Phil. Trans. Roy. Soc*, 1860, p. 133.

[4] Riemann, Werke, p. 157. *Gott. Abh.*, t VIII, p 43 (1858-59).

[5] Rankine, *Mis. Sc. Papers*, p. 530, *Phil. Trans.*. Vol 160, p. 277, 1870

[6] Hugoniot, *Journ. de l' Ecole Polytechnique* (1887).

[7] Rayleigh, 'Aerial Plane waves of Finite Amplitude,' *Proc. Roy. Soc.*, A, Vol. 84 pp. 247-284 (1910).

As in the case of plane waves the same difficulty will of course be experienced on account of the motion becoming ultimately discontinuous. For a detail discussion of the question of discontinuity for plane waves reference may be made to Taylor [1] and Wilton. [2]

That the ordinary approximate theory of infinitesimal aerial waves is incapable of explaining many of the phenomena observed with waves of great intensity such as that which accompanies a sudden explosion, a gun-fire or an electric discharge has been long recognised by the physicists. Quite recently Dr. Louis Vesot King [3] has given an interesting application of the theory of plane waves of finite amplitude in a discussion of the acoustic efficiency of fog-signal machinery. In a subsequent paper, as an application of the theory of spherical waves of finite amplitude, a discussion will be given of the explosion waves arising from the sudden explosion of a detonating gas contained within a spherical envelope.

The Eulerian equations for spherical disturbances are

$$\frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial r}=-\frac{1}{\rho}\,\frac{\partial p}{\partial r}, \qquad \ldots \quad (1)$$

$$\frac{\partial \rho'}{\partial t}+u\,\frac{\partial \rho'}{\partial r}=-\frac{\partial u}{\partial r},$$

where $\rho'=\log \rho r^2$, ρ denoting the density.

The equations when written in this form are remarkably similar to the equations for plane waves of finite amplitude, *viz.*,

$$\frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial x}=-\frac{1}{\rho}\,\frac{\partial p}{\partial x},$$

$$\frac{\partial \rho_1}{\partial t}+u\,\frac{\partial \rho_1}{\partial x}=-\frac{\partial u}{\partial x},$$

where $\rho_1=\log \rho$, ρ being the density.

To determine the laws of propagation, we use Riemann's method. If we put

$$P=f(\rho')+u, \quad Q=f(\rho')-u,$$

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[1] Taylor, *Proc. Roy. Soc.*, 1910.

[2] Wilton, *Phil. Mag.*, 1913

[3] Louis Vesot King, "On the Propagation of Sound in the free atmosphere and the acoustic efficiency of fog-signal machinery, *Phil. Trans.*, Vol. 218, 1919.

where $f(\rho')$ is as yet undetermined, we obtain by differentiation

$$\frac{\partial P}{\partial t}=f'(\rho')\frac{\partial \rho'}{\partial t}+\frac{\partial u}{\partial t},$$

$$\frac{\partial P}{\partial r}=f'(\rho')\frac{\partial \rho'}{\partial r}+\frac{\partial u}{\partial r};$$

$$\frac{\partial Q}{\partial t}=f'(\rho')\frac{\partial \rho'}{\partial t}-\frac{\partial u}{\partial t},$$

$$\frac{\partial Q}{\partial r}=f'(\rho')\frac{\partial \rho'}{\partial r}-\frac{\partial u}{\partial r}.$$

Therefore

$$\frac{\partial P}{\partial t}+u\frac{\partial P}{\partial r}=\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)+f'(\rho')\left(\frac{\partial \rho'}{\partial t}+u\frac{\partial \rho'}{\partial r}\right)$$

$$=-\frac{1}{\rho}\frac{\partial p}{\partial r}-f'(\rho')\frac{\partial u}{\partial r}.$$

If now we determine $f(\rho')$ so that

$$\{f'(\rho')\}^2\frac{\partial \rho'}{\partial r}=\frac{1}{\rho}\frac{\partial p}{\partial r},$$

we get

$$\frac{\partial P}{\partial t}+u\frac{\partial P}{\partial r}=-\left[f'(\rho')\frac{\partial \rho'}{\partial r}+\frac{\partial u}{\partial r}\right]f'(\rho')$$

$$=-f'(\rho')\frac{\partial P}{\partial r}.$$

Similarly,

$$\frac{\partial Q}{\partial t}+u\frac{\partial Q}{\partial r}=f'(\rho')\left(\frac{\partial \rho'}{\partial t}+u\frac{\partial \rho'}{\partial r}\right)-\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)$$

$$=-f'(\rho')\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$=f'(\rho')\left[f'(\rho')\frac{\partial \rho'}{\partial r}-\frac{\partial u}{\partial r}\right]$$

$$=f'(\rho')\frac{\partial Q}{\partial r}.$$

Thus we obtain

$$dP=[dr-\{f'(\rho')+u\}dt]\,\frac{\partial P}{\partial r},$$

$$dQ=[dr+\{f'(\rho')-u\}dt]\,\frac{\partial Q}{\partial r}.$$

Hence in spherical waves of finite amplitude $dP=0$, or P is constant, for a geometrical point moving with the velocity

$$\frac{dr}{dt}=f'(\rho')+u$$

$$=\left[\frac{1}{\rho}\,\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}+u, \qquad \dots \quad (2)$$

whilst Q is constant for a point moving with the velocity

$$\frac{dr}{dt}=-f'(\rho')+u$$

$$=-\left[\frac{1}{\rho}\,\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}+u. \qquad \dots \quad (3)$$

Hence any given value of P moves forward, and any value of Q moves backward, with the velocity given by (2) or (3) as the case may be.

Now since $\left[\frac{1}{\rho}\,\frac{dp}{d\log(\rho r^2)}\right]$ can also be written in the form

$$\left[\frac{1}{\rho}\,\frac{dp}{d\rho}\,\frac{1}{\frac{1}{\rho}+\frac{2}{r}\frac{dr}{d\rho}}\right],$$

the above expressions for the velocities of propagation for the forward and the backward motion will reduce when r is very large to the well-known expressions

$$\left(\frac{dp}{d\rho}\right)^{\frac{1}{2}}+u \text{ and } -\left(\frac{dp}{d\rho}\right)^{\frac{1}{2}}+u$$

for these quantities for plane waves of finite amplitude just as we should expect.

As in the case of Riemann's method for plane waves of finite amplitude, the above results well enable us to understand in a general way, the nature of the motion in any given case. Thus if the initial disturbance be supposed to be confined between the two spherical surfaces $r=a$, $r=b$, we may suppose that P and Q both vanish for $r>a$ and for $r<b$. The region within which P is variable will advance, and that within which Q is variable will recede, until after a time these regions separate and leave between them a space for which $P=0$, $Q=0$ and in which the fluid is therefore at rest. The original disturbance has thus been split up into a diverging and a converging wave travelling in opposite directions. In the advancing wave we have $Q=0$ and therefore $u=f(\log \rho r^2)$. It ill thus appear that in spherical waves of finite amplitude both the density and the particle velocity are not propagated forwards at the same rate as in the case of the plane waves of finite amplitude. The velocity of propagation will however be found to be greater, the greater the value of ρ. If we draw a curve with r as abscissa and ρ as ordinate, and make each point of this curve move forward with the appropriate velocity, those parts of the curve will move faster which have the greater ordinates and the curve will eventually become at some points perpendicular to r making the quantities $\frac{du}{d\rho}$, $\frac{d\rho}{dr}$ infinite and giving rise to the well known case of the wave of discontinuity.

It is interesting to trace the variation of the velocity of propagation at different distances from the source. For this purpose it is convenient to write the quantity $\frac{1}{\rho}\frac{dp}{d\log \rho r^2}$ in the form $\frac{dp}{d\rho}\cdot\frac{1}{1+2A\rho/r}$ where $A=\frac{dr}{d\rho}$.

Whether at any point the velocity of propagation so far as it is dependant on this quantity is less or greater than the *standard* velocity $\left(\frac{dp}{d\rho}\right)^{\frac{1}{2}}$ obviously depends on the sign of A, that is, on the sign of $\frac{d\rho}{dr}$. If the wave be supposed to start from a spherical surface of small radius a as a wave of condensation followed by rarefaction, then initially $\frac{d\rho}{dr}$ is negative and the initial velocity is greater than the standard velocity. If on the other hand it starts as a wave of rarefaction followed by condensation, the initial velocity is less than the standard velocity. If discontinuity does not set in at any stage of the motion, then the velocity of propagation approaches the standard velocity at a considerable distance from the origin.

Using the Lagrangian method if ξ denotes the displacement at time t of the particle whose undisturbed abscissa is r and if we write

$y=r+\xi$, so that y denotes the absolute coordinate at time t, the dynamical equation obtained by considering the forces acting on a unit area of a stratum formed by two consecutive spherical surfaces is

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = -\frac{\partial p}{\partial r},$$

where ρ_0 is the density in the undisturbed state and the equation of continuity is

$$\rho_0 r^2 = \rho y^2 \frac{\partial y}{\partial r},$$

ρ being the density in the disturbed state.

If as before we write $\rho' = \log \rho y^2$, and if we suppose that the pressure is a function of ρ only, the dynamical equation can be written in the form

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = -\frac{dp}{d\rho'} \cdot \frac{\partial \rho'}{\partial r},$$

Also since $\rho' = \log \rho y^2$, we have $\frac{\partial \rho'}{\partial r} = \frac{1}{\rho y^2} \frac{\partial (\rho y^2)}{\partial r}$. Also from the equation of continuity $\rho_0 = 3\rho y^2 \frac{\partial y}{\partial r^3}$ and therefore $-\frac{\rho_0}{3(\rho y^2)^2} \frac{\partial (\rho y^2)}{\partial r}$ $= \frac{\partial^2 y}{\partial r \partial r^3}$.

Consequently
$$-\frac{\rho_0}{3\rho y^2} \frac{\partial \rho'}{\partial r} = \frac{\partial^2 y}{\partial r \partial r^3}.$$

The dynamical equation can therefore be written in the form

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = \frac{dp}{d\rho'} \cdot \frac{3\rho y^2}{\rho_0} \frac{\partial^2 y}{\partial r \partial r^3} = \frac{dp}{d\rho'} \cdot \frac{\frac{\partial^2 y}{\partial r \partial r^3}}{\frac{\partial y}{\partial r^3}}.$$

That is to say,

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{\rho} \frac{dp}{d(\log . \rho y^2)} \cdot \frac{\frac{\partial^2 y}{\partial r^3 \partial r^3}}{\frac{y^2}{r^2}\left(\frac{\partial y}{\partial r^3}\right)^2}.$$

It is easy to express the differential equation also in the form

$$\frac{\partial^2 y}{\partial t^2} = \frac{dp}{d\rho} \cdot \frac{\frac{\partial}{\partial r}\left(\frac{y^2}{r^2} \frac{\partial y}{\partial r}\right)}{\frac{y^4}{r^4}\left(\frac{\partial y}{\partial r}\right)^2} \quad \ldots \quad (4)$$

that is to say in the form

$$\frac{\partial^2 y}{\partial t^2}=-\frac{dp}{d\rho}\frac{\partial}{\partial r}\left[\frac{1}{\frac{y^2}{r^2}\frac{\partial y}{\partial r}}\right].$$

A general solution of this differential equation is not easy to obtain An asymptotic solution can however be easily obtained. At a great distance from the origin, that is say, when r is large, this equation reduces to the form

$$\frac{\partial^2 y}{\partial t^2}=\frac{dp}{d\rho}\cdot\frac{\frac{\partial^2 y}{\partial r^2}}{\left(\frac{\partial y}{\partial r}\right)^2}$$

which is of the same form as the Earnshaw's differential equation. The general integral of this latter equation is obtained by eliminating a between the equations

$$y=ar+(C\pm c\log a)t+\phi(a)$$

$$0=ar\pm ct+a\phi'(a),$$

where C is a constant, ϕ is arbitrary and $c^2=\frac{dp}{d\rho}$. This is as we should expect for the spherical disturbances ultimately vanish as a set of plane waves.

Earnshaw's method can be adopted to obtain a general integral of this equation if it is possible to obtain a function τ. such that

$$\frac{\partial y^3}{\partial \tau}=\frac{1}{\sqrt{3}r}\frac{\partial y}{\partial t},$$

τ being independent of y.

For writing the differential equation in the form

$$\frac{1}{3r^2}\frac{\partial^2 y}{\partial t^2}=\frac{dp}{d\rho}\cdot\frac{\frac{\partial^2 y^3}{\partial (r^3)^2}}{\left(\frac{\partial y^3}{\partial r^3}\right)^2}$$

and making the assumption

$$\frac{\partial y^3}{\partial \tau}=F\left(\frac{\partial y^3}{\partial r^3}\right),$$

we easily obtain the first integral

$$\frac{\partial y^3}{\partial \tau} = C \pm c \log \frac{\partial y^3}{\partial r^3} = \frac{u}{\sqrt{3}r},$$

and the general integral is obtained by eliminating a between the two equations

$$y^3 = ar^3 + \frac{u}{\sqrt{3}r}\tau + \phi(a)$$

$$0 = ar^3 \pm c\tau + a\phi'(a),$$

where as before $c^2 = \frac{dp}{d\rho}$ and ϕ is arbitrary.

On the Theory of continued Fraction

[Third Paper]

By

HARIPADA DATTA.

[Read September 4th, 1920]

When the series

$$\frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \dots$$

is converted into a continued fraction of the form

$$\frac{a_1}{x+b_1} + \frac{a_2}{x+b_2} + \frac{a_3}{x+b_3} + \dots, \qquad \dots (1)$$

we know that the elements of this continued fraction are

$$a_1 = a_1,$$

$$a_n = \frac{K_n K_{n-2}}{K^2_{n-1}},$$

and

$$b_n = \frac{{}^1K_{n-1}}{K_{n-1}} - \frac{{}^1K_n}{K_n},$$

where

$$K_{n+1} = \begin{vmatrix} a_{n+1} & a_n & \dots & a_2 & a_1 \\ a_{n+2} & a_{n+1} & \dots & a_3 & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{2n} & a_{2n-1} & \dots & a_{n+1} & a_n \\ a_{2n+1} & a_{2n} & \dots & a_{n+2} & a_{n+1} \end{vmatrix}$$

By deleting the $(r+1)$th row and the last column we obtain ${}^rK_n$.

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[1] *This paper was written on the 16th of October 1917, when I was working in Cambridge as a Carniegi Research Student.*

1. With the help of the above theorem the series

$$\frac{a_1}{x}+\frac{a_3}{x^3}+\frac{a_7}{x^7}+\ldots+\frac{a_{2^n-1}}{x^{2^n-1}}+\quad\ldots\quad(2)$$

is to be converted into a continued fraction.

As in the series (2)

$$a_2 = a_4 = a_5 = a_6 = a_8 = a_9 = \ldots = 0$$

and it can be shown that

$$\left.\begin{aligned}
K_{2^n} &= -a_{2^{n+1}-1} K_{2^n-1}\\
K_{2^n+1} &= a^{3}_{2^{n+1}-1} K_{2^n-2}\\
K_{2^n+2} &= -a^{5}_{2^{n+1}-1} K_{2^n-3}\\
\ldots &\qquad \ldots \qquad \ldots\\
K_{2^{n+1}-2} &= -a^{2^{n+1}-3}_{2^{n+1}-1} K_1\\
K_{2^{n+1}-1} &= -a^{2^{n+1}-1}_{2^{n+1}-1} K_0
\end{aligned}\right\}$$

where $K_0 = 1, n = 1, 2, 3, \ldots$ and $2^n + s \not> 2^{n+1}-1$.

From these we can find the a's in terms of a's. All the b's are zeros, for all the determinants of the form ${}^1K_n$ are zeros.

Making all the a's except a_1 negative we obtain

$$\frac{a_1}{x}-\frac{a_3}{x^3}-\frac{a_7}{x^7}-\ldots$$

$$=\frac{a_1}{x+}\,\frac{c_2}{x+}\,\frac{c_3}{-x+}\,\frac{c_4}{x+}\,\frac{c_5}{x+}\,\frac{c_6}{-x+}\,\frac{c_7}{-x+}\ldots,\quad\ldots\quad(3)$$

where the 2^nth x, $(n=1, 2, \ldots)$ that is the denominator of the 2^n part-fraction or link in the chain, of the continued fraction (3) is positive; and the (2^n-1) x's on the right of the central x are opposite in signs to the corresponding x's on the left of it. All the c's are positive. The (2^n-2) c's on the right of c_{2^n+1} are equal to the corresponding c's on the left of c_{2^n}; and $c_{2^n} = c_{2^n+1}$. Thus if we know the denominators and the numerators of the first two part-fractions and also the numerators of the 2^nth part-fraction ($n=1, 2, \ldots$) then we can write down the whole continued fraction (3)

As $c_2 = a_3/a_1$; and $c_{2^n} = a_1\, a_{2^{n+1}-1} \big| a^2_{2^n-1}$.

Therefore

$$\frac{a_1}{x} - \frac{a_3}{x^3} - \frac{a_7}{x^7} - \ldots$$

$$= \frac{a_1}{x} + \frac{a_3/a_1}{x} + \frac{a_3/a_1}{-x} + \frac{a_1\, a_7/a_3^2}{x} + \frac{a_3/a_1}{-x} + \frac{a_3\ a_1}{-x} + \ldots \quad (4)$$

If we consider the convergents in connection with the series and express the denominators as polynomials in x whose coefficients are the functions of the coefficients in the series then it will appear that if the n terms next to the nth term of the power-series

$$\frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \ldots$$

are absent then the nth convergent will be a finite series, that is, the first n terms of the series. Thus in the case of the continued fraction (4) the (2^n-1)th convergents are all finite series. In the case of the continued fraction (4) this can be shown, without the intervention of the series, in the following way.

3. Let

$$\mathrm{D}_7 = \begin{vmatrix} b_1 & a_2 & & & & & \\ 1 & b_2 & a_3 & & & & \\ & 1 & b_3 & a_4 & & & \\ & & 1 & 0 & a_4 & & \\ & & & 1 & -b_3 & a_3 & \\ & & & & 1 & -b_2 & a_2 \\ & & & & & 1 & -b_1 \end{vmatrix} = (-1)^7\, \mathrm{D}_7.$$

Therefore $D_7 = 0.$

If the zero in the centre of the continuant be replaced by x_1, and this new continuant be denoted by N_7, then it can be shown that

$$N_7 = D_7 + (\mp x_1)^4 \begin{vmatrix} b_1 & a_2 & \\ 1 & b_2 & a_3 \\ & 1 & b_3 \end{vmatrix} \begin{vmatrix} -b_3 & a_3 & \\ 1 & -b_2 & a_2 \\ & 1 & -b_1 \end{vmatrix}$$

and therefore

$$= -x_1 \begin{vmatrix} b_1 & a_2 & \\ 1 & b_2 & a_3 \\ & 1 & b_3 \end{vmatrix}^2$$

If $b_3 = -b_1$; $b_2 = x_2$, and $a_2 = a_3$, then $N_7 = -x_1 x_2^2 b_1^2$.

Thus we see that the denominator of the (2^n-1)th convergent to the continued fraction (4)

$$= -x \cdot x^2 \cdot x^4 \ldots x^{2^{n-1}} = -x^{2^n-1}.$$

Hence the (2^n-1)th convergent is a finite series consisting of the first (2^n-1) terms (zero-coefficients being included) of the series.

4. When a finite series is converted into a continued fraction of the form (1) as well as into another form of continued fraction known as the Equivalent continued fraction, both these continued fractions are finite and equal to each other. Therefore there must be some method or methods of transforming one form of continued fraction into the other. In the case of the continued fraction (4), the transformation can be effected, owing to the presence of the zero-coefficients in the series, in the following way :—

Let us take the continued fraction of the first (2^n-1) part fractions. Leaving aside the first part fraction, if we contract the rest of this finite continued fraction, we obtain a continued fraction of 2^{n-1} part—fractions. Again leaving aside the first two links of this transformed continued fraction and contracting the rest of it, we obtain a continued fraction of $2^{n-2}+1$ links. In this way the continued fraction can be reduced to n part-fractions and which, as a matter of fact, is the equivalent continued fraction, As for example, the seventh convergent to the continued fraction (4), when thus contracted becomes

$$\frac{a_1}{x+}\,\frac{a_5 x}{a_1 x^2 - a_5 -}\,\frac{a_1 a_7 x^2}{a_3 x^4 + a_7}.$$

Thus we see though the continued fraction (4) belongs to the form (1), yet it can be regarded as an extended form of the equivalent continued fraction. It is to be noted also that the convergents other than the (2^n-1) th convergent $(n=1, 2\dots)$ are not finite series.

5. As a general case of (2) we can show that

$$\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\dots+\frac{a_r}{x^r}+\frac{a_{2r+1}}{x^{2r+1}}+\frac{a_{4r+3}}{x^{4r+3}}+\frac{a_{8r+7}}{x^{8r+7}}+\dots$$

$$=\frac{a_1}{x+b_1+}\;\frac{a_2}{x+b_2+}\;\frac{a_3}{x+b_3+\dots}\;\frac{a_r}{+x+b_r+\dots}\;\frac{a_{r+1}}{x+}\;\frac{a_{r+1}}{-(x+b_r)+}$$

$$\frac{a_r}{-(x+b_{r-1})+\dots}\;\frac{a_2}{-(x+b_1)+}\;\frac{a_{2r+2}}{x+}\;\frac{a_{2r+2}}{x+b_1+}\;\frac{a_2}{x+b_2+\dots}$$

$$\dots\frac{a_{r+1}}{x+b_{r-1}+}\;\frac{a_{r+1}}{x+b_r+}\;\frac{a_r}{-x+}\;\frac{a_{r-1}}{-(x+b_r)+\dots}\;\frac{a_2}{-(x+b_1)+}$$

$$\frac{a_{4r+4}}{x+}\;\frac{a_{4r+4}}{x+b_1+}\;\frac{a_2}{x+b_2+\dots}$$

Here any two quantities $a_2, a_3,\dots a_r$ are not necessarily equal to one another, and also $b_1, b_2,\dots b_r$ are not necessarily zeros. If we know the denominators and the numerators of the first r part—fractions as well as the numerator of the $2^n(r+1)$ th part fraction $(n=0, 1, 2\dots)$ then we can write down the whole of this continued fraction. As for example

$$\frac{1}{x}-\frac{2}{x^2}-\frac{3}{x^3}-\frac{\alpha}{x^7}-\frac{\beta}{x^{15}}-\dots$$

$$=\frac{1}{x-2+}\;\frac{1}{x-4+}\;\frac{27}{x+6+}\;\frac{\alpha/27}{-(x+6)+}\;\frac{27}{4-x+}\;\frac{1}{2-x+}$$

$$\frac{\beta/\alpha^2}{x+}\;\frac{\beta/\alpha^2}{x-2+}\;\frac{1}{x-4+}\;\frac{27}{x+6+}\;\frac{\alpha/27}{-x+}\;\frac{27}{4-x+}\;\frac{1}{2-x+\dots}$$

6. The series

$$\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\dots$$

may also be converted into a continued fraction of the form

$$\frac{c_1}{x+}\;\frac{c_2}{1+}\;\frac{c_3}{x+}\;\frac{c_4}{1-}\quad\ldots\qquad(5)$$

where
$$c_{2n} = -\frac{K_{n-1}}{K_n}\cdot\frac{{}^{n}K_n}{{}^{n-1}K_{n-1}},$$

$$c_{2n+1} = \frac{K_{n+1}}{K_n}\cdot\frac{{}^{n-1}K_{n-1}}{{}^{n}K_n}.$$

The continued fraction (5) is the extended from of (1), for the $2n$ th convergent of (5) is the same as the n th covergent of (1).[1]

If we convert the series (2) into a continued fraction of the form (5), we find that some of the determinants of the type ${}^{n}K_n$ vanish, hence some of the quantities c's become zeros or infinities. Though this is the case, *still the even convergents of (5) do possess definite values and the continued fraction (5) is not meaningless.*

Now we shall enquire into the case where c_{2n} is zero and none of the other c's are zero or infinity. c_{2n} can become zero if either K_{n-1} or ${}^{n}K_n$ vanishes. In this case K_{n-1} cannot vanish therefore ${}^{n}K_n$ does vanish. As the c's with suffixes greater than $2n$, are neither zeros nor infinities and therefore

$$K_n = K_{n+1} = \ldots = 0,\qquad\ldots\quad(6)$$

$${}^{n}K_{n+1} = {}^{n+1}K_{n+2} = \ldots = 0.\qquad\ldots\quad(7)$$

Now we shall show that if (6) and (7) hold then any convergent of order higher than $2n-1$, is equal to $(2n-1)$ th convergent.

7. Let $$K_3 = K_4 = 0,$$

we know that

$$K_3\begin{vmatrix} a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \\ a_4 & a_5 & a_7 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_4 & a_5 & a_6 \end{vmatrix}^2 = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} K_4.$$

---

[1] *If the series* $\frac{a_1}{y^2}+\frac{a_2}{y^4}+\frac{a_3}{y^6}+\ldots$ *be converted into a continued fraction, of the form (1), this continued fraction owing to the absence of the odd powers of* $\frac{1}{y}$ *in the series will be the same as the continued fraction* (5).

Therefore

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_4 & a_5 & a_6 \end{vmatrix}=0.$$

Hence we obtain

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{Vmatrix}=0. \qquad \ldots \quad (8)$$

If also $K_5=0$,

then we have

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ a_4 & a_5 & a_6 & a_7 & a_8 \end{Vmatrix}=0.$$

Hence,

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \\ a_5 & a_6 & a_7 & a_8 \end{vmatrix}=0 \qquad \ldots \quad (9)$$

and also from (8) we have

$$\begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ a_4 & a_5 & a_6 \end{vmatrix}=0. \qquad \ldots \quad (10)$$

From (9) and (10) we obtain

$$\begin{Vmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{Vmatrix}=0. \qquad \ldots \quad (11)$$

Now combining (11) with (8) we find that

$$\left\|\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 & a_7 \end{matrix}\right\| = 0.$$

Similarly if (6) holds then

$$\left\|\begin{matrix} a_1 & a_2 & \dots \\ a_2 & a_3 & \dots \\ a_{n+1} & a_{n+2} & \dots \end{matrix}\right\| = 0 \qquad \dots \quad (12)$$

It has been shown in a previous paper[1] that if (12) holds good the even convergents of order higher than $2n$, are all equal to the $2n$th convergent. All the even cenvergents (5) can be connected by recurrence formulae as has been done in (11), and so also the odd convergents. Similarly it can be shown from (7) that any odd convergent of order higher then $(2n-1)$ is equal to the $(2n-1)$th convergent. Exactly in a similar manner it can be shown that if c_n vanishes (whether it is due to the vanishing of K_r or ${}^rK_r$) and none of the other c's is zero or infinity then the continued fraction will terminate at $(n-1)$th convergent.

8. The series

$$\frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_4}{x^4} + \frac{a_8}{x^8} + \frac{a_{16}}{x^{16}} + \dots + \frac{a_{2^n}}{x^{2^n}} + \dots$$

is to be converted into a continued fraction of the form (1).

Here we have

$$\left.\begin{aligned} K_{2^n} &= a_{2^n}^{2^n} K_0 \\ K_{2^n-1} &= a_{2^n}^{2^n-2} K_1 \\ K_{2^n-2} &= a_{2^n}^{2^n-4} K_2 \\ &\dots \\ K_{2^{n-1}+1} &= a_{2^n}^{2} K_{2^{n-1}-1} \end{aligned}\right\}$$

---

[1] On the Failure of Heilermann's Theorem *Proc. Edinburgh Math. Soc.* Vol. XXXV (Part 2), 1916-17.

$$\left.\begin{aligned}
{}^1K_{2^n} &= -a_{2^{n+1}}\,{}^1K_{2^{n-1}} \\
{}^1K_{2^n-1} &= -a_{2^n}^{2^n-2}\,{}^1K_1 \\
{}^1K_{2^n-2} &= -a_{2^n}^{2^n-4}\,{}^1K_2 \\
\cdots\quad & \quad\cdots \\
{}^1K_{2^n+1} &= -a_{2^n}^{2}\,{}^1K_{2^{n-1}-1}.
\end{aligned}\right\}$$

From these we find that the continued fraction is

$$\frac{\alpha_1}{x-\frac{\alpha_2}{\alpha_1}+}\;\frac{a_2}{x+b_2+}\;\frac{a_3}{x+b_3+}\;\frac{a_2}{x+b_4+}\;\frac{a_5}{x+b_5+}$$

$$\frac{a_2}{x+b_3+}\;\frac{a_3}{x+b_2+}\;\frac{a_2}{x+b_8+}\;\frac{a_9}{x+a_9+}\cdots$$

where

$$a_2=\left(\frac{\alpha_1\alpha_2}{\alpha_1}\right)^2 \qquad b_4=-\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1\alpha_8}{\alpha_4^2}$$

$$a_3=\left(\frac{\alpha_1\alpha_4}{\alpha_4^2}\right)^2 \qquad b_5=-\left(\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1\alpha_8}{\alpha_4^2}\right)$$

$$a_5=\left(\frac{\alpha_1\alpha_8}{\alpha_4^2}\right)^2 \qquad b_8=-\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1\alpha_{16}}{\alpha_8^2}$$

$$a_9=\left(\frac{\alpha_1\alpha^{16}}{\alpha_8^2}\right) \qquad b_9=-\left(\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1\alpha_{16}}{\alpha_8^2}\right)$$

$$b_2=\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1\alpha_4}{\alpha_2^2} \qquad b_{16}=-\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1\alpha_3^2}{\alpha_{16}^2}$$

$$b_3=\frac{\alpha_2}{\alpha_1}-\frac{\alpha_1\alpha_4}{\alpha_2^2} \qquad b_{17}=-\left(\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1\alpha_{32}}{\alpha_{10}^2}\right).$$

From these the laws existing among the denominators and the numerators as regards their arrangement, may be deduced.

Notes and News

At the meeting of the Society held on the 5th April, 1920, Prof. C. V. Raman moved a resolution congratulating the Hon'ble Sir Asutosh Mookerjee, Kt., C.S.I., the President of the Society, on his being appointed the Officiating Chief Justice of Bengal, a position which he so eminently deserved. The resolution was seconded by Mr. D. N. Mitra and was carried with acclamation.

The Society received with deep regret the news of the death of the distinguished Indian Mathematician Mr. S. Ramanujan, B.A., F.R.S., in the prime of his life. Indian Mathematics has sustained almost an irrepairable loss by the untimely death of this Indian 'genius.' One fact must have struck those who have carefully studied the life of Mr. Ramanujan and it is this that with proper opportunities Indians can rise to the most eminent position in Science. Those who have any knowledge of the children of the soil will readily admit that a young chap showing wonderful precocity in mathematics is not a too infrequent occurrence in this country. But how many of these would-be genius retain their aptitude for mathematics and are not simply nipped in the bud for want of opportunities !

An International Congress of Mathematicians will be held in Strasbourg on September 22, 1920. The following officers have been elected by the National Committee of French Mathematicians for organising the Congress in Strasbourg, President d'Honneur : M. Jordan ; President : M. Picard ; Vice-Presidents : MM. Appel, Borel, Lecornu, Le Roux ; General Secretary : M. Koenigs ; Secretary : M. Gilburn ; Treasurer : M. Miluski. The Society has been invited to participate in the function. We wish the Congress a complete success.

The following courses of instruction to be delivered in the University of Strasbourg is published for the information of our students :

Studies in the Institut de Mathématiques de L'Université de Strasbourg.

Helped by the favourable exchange level, a number of students are coming to study in French Universities. It is important to bring to their attention that, since November, 1919, the University of

Strasbourg, completely reorganised, is working in full order. Its teaching staff is more than equal in number to what it was under German rule, and its equipment, already excellent in many respects has been greatly improved where it was deficient.

Concerning mathematical teaching, young students will be offered in Strasbourg the usual standard courses on Analysis, Mechanics, Astronomy ... whose programme is permanent and should require the students' activity for two or three years. Furthermore, a set of *research* courses have been arranged for the use of candidates to the "Doctorat de l'Université de Strasbourg," and of scholars generally (French diplomas are required for registration in view of the doctorship, but can be dispensed with on production of equivalent foreign diplomas, with appreciation of the students' ability by one of his former Professors).

Programme of *research* courses during the academic year 1920-1921.

First Semester (1st November, 1920 to 28th February, 1921).

Mathematical Physics—Mr. Bauer ...	Quanta's theory; Atoms' structure: 3 lectures a week.
Higher Analysis—Mr. Fréchet ...	Theory of chance: 2 weekly lectures; Integral equations: 1 weekly lecture.

Second Semester (1st March, 1921 to 30th June, 1921).

Mathematical Physics—Mr. Bauer ...	Statistical applications of Quanta's theory: 3 weekly lectures.
Higher Analysis—Mr. Fréchet ...	Applications of the theory of chance: 1 weekly lecture. Functions of lines: 2 weekly lectures.
Hydrodynamics—Mr. Villat ...	Researches on a solid's movement in a viscous fluid: 2 weekly lectures.
Differential Geometry—Mr. Pérès ...	Transformations of surfaces applicable on quadrics: 2 weekly lectures.
Theory of Functions—Mr. Valiron ...	Dirichlet's series and faculties series: 2 weekly lectures.

For further information apply (in French or English) to : M. le Directeur de l'Institut de Mathematiques de Strasbourg, Bas-Rhin, France. Details concerning lodgings etc. will be supplied by the Comité de Patronage des studiants étrangers, Université, Strasbourg, Bas-Rhin France. Students who want to improve their knowledge of the French language during the vacation may apply to the 1st address for the tract on "Summer Courses," organized by the "Faculté des Lettres de Strasbourg" to meet their wishes.

Mr. Oudh Upadhyaya of Benares gave a fair example of the nature of the work done by the newly started Mathematical Society at Benares at the meeting of the Calcutta Mathematical Society held on the 5th April, 1920. He strongly criticised a paper "On certain new Tautochrones, determinable by quadratures" published by Mr. Shukdeo Pande of Benares in the only volume which has hitherto been published by the Benares Society. He pointed out that any curve will be a tautochrone if the law of tangential force be proportional to the length of the arc measured from a certain fixed point on the curve. For a rectifiable curve the law of force can obviously be very easily determined. This leads Mr. Upadhyaya to think that all the results of Mr. Pande must have been obtained from the 'known results for rectifiable curves by a process of backward calculation.'

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Evaluation of the product matrix in a commutantal product of simple matrices having given nullities.

BY

C. E. CULLIS.

[*Read, September 4th, 1920*]

[*Summary.* The object of this paper is the exact determination of every element of the product matrix in a commutantal product of simple matrices in each of which a given number of horizontal rows, a given number of vertical rows, and a given number of diagonal lines, counting from the base in each case, contain only zero elements. Products of four matrices only are considered, these being sufficiently typical of products of any number of matrices. The final results are embodied in the theorems of Art. 7, and contain as particularisations the most important properties of many special kinds of simple matrices. The other articles define and interpret the notations and the nomenclature, and lead up to the general theorems. Readers to whom the meaning of a 'commutantal product' is not clear may suppose in the first instance that all matrices have their natural types, and then make the obvious generalisations.]

1. Standard notations for simple matrices having assigned types.

1. *Simple matrices having assigned non-ruled types.*

By the four *direct* standard general simple matrices $X=[x]_m^n$ of the class $M\binom{n}{m}$ and of the 1st, 2nd, 3rd, 4th of the non-ruled types

$$\theta_1=\{\pi, \pi\},\ \theta_2=\{\pi', \pi'\},\ \theta_3=\{\pi', \pi\},\ \theta_4=\{\pi, \pi'\} \qquad \dots \quad (a)$$

will be meant the 1st, 2nd, 3rd, 4th of the matrices

$$\overset{\longrightarrow}{\begin{bmatrix} x_{11} \dots x_{1n} \\ \dots \quad \dots \\ x_{m1} \dots x_{mn} \end{bmatrix}},\quad \underset{\longleftarrow}{\begin{bmatrix} x_{mn} \dots x_{m1} \\ \dots \quad \dots \\ x_{1n} \dots x_{11} \end{bmatrix}},\quad \underset{\longrightarrow}{\begin{bmatrix} x_{m1} \dots x_{mn} \\ \dots \quad \dots \\ x_{11} \dots x_{1n} \end{bmatrix}},\quad \overset{\longleftarrow}{\begin{bmatrix} x_{1n} \dots x_{11} \\ \dots \quad \dots \\ x_{mn} \dots x_{m1} \end{bmatrix}} \qquad \dots \quad (A)$$

in which the elements are arbitrary parameters. These are a complete set of simple matrices correlated by simple reversals; and by assigning the type θ_1 to the first of them, we fix the types of the other three. When we assign a non-ruled type θ to a simple matrix $[x]^{n}_{m}$ and say that it is expressed in the standard direct form appropriate to that type, we regard it as a particularisation of the direct standard general simple matrix of the same class and type as defined above, the same notations being used for its elements.

By suitably arranging the conjugates of the four matrices (A) we obtain the four *inverted* standard general simple matrices $X = \underset{\llcorner\lrcorner n}{\overset{\ulcorner\urcorner m}{x}}$ of the class $M\binom{m}{n}$ and of the 1st, 2nd, 3rd, 4th of the types (a), which we define to be the 1st, 2nd, 3rd, 4th of the matrices

$$\begin{bmatrix} x_{mn} \cdots x_{1n} \\ \cdots \quad \cdots \\ x_{m1} \cdots x_{11} \end{bmatrix} \begin{array}{c} \downarrow \\ \uparrow \end{array}, \quad \begin{bmatrix} x_{11} \cdots x_{m1} \\ \cdots \quad \cdots \\ x_{1n} \cdots x_{mn} \end{bmatrix}, \qquad \begin{bmatrix} x_{m1} \cdots x_{11} \\ \cdots \quad \cdots \\ x_{mn} \cdots x_{1n} \end{bmatrix} \begin{array}{c} \downarrow \\ \uparrow \end{array}, \quad \begin{bmatrix} x_{1n} \cdots x_{mn} \\ \cdots \quad \cdots \\ x_{11} \cdots x_{m1} \end{bmatrix} \qquad \text{(B)}$$

in which the elements are arbitrary parameters. These are also a complete set of simple matrices correlated by simple reversals; and by assigning the type θ, to the first of them, we fix the types of the other three. When we assign a non-ruled type θ to a simple matrix $\underset{\llcorner\lrcorner n}{\overset{\ulcorner\urcorner m}{x}}$, and say that it is expressed in the standard inverted form appropriate to that type, we regard it as a particularisation of the inverted standard general simple matrix of the same class and type as defined above, the same notations being used for its elements.

By interchanging m and n in the four matrices (B) we obtain the corresponding four inverted standard general simple matrices $X = \underset{\llcorner\lrcorner m}{\overset{\ulcorner\urcorner n}{x}}$ of the class $M\binom{n}{m}$. Thus we have defined eight standard general simple matrices, four direct and four inverted, in every class.

In each of the eight matrices (A) and (B), whether general or particularised, we will call

x_{11} the *first* or *initial parametric element*,

x_{mn} the *last* or *final parametric element*,

x_{1n} the *apical element*, and the corner occupied by x_{1n} the *apex*,

x_{m1} the *basical element*, and the corner occupied by x_{m1} the *base*.

The initial element is marked by an arrow, the apex by an arrow head, and the base by a slanting line, which (see Art. 2) is parallel to the parametric diagonal lines, whilst the final element is left unmarked. In each of the four matrices (A) having the direct forms the initial element lies in the same horizontal line as the apex, and in each of the four matrices (B) having the inverted forms it lies in the same vertical line as the apex. The one arbitrary choice of type in (B) is so made that in every standard general simple matrix the positions of the apex and base are fixed by the type. There are eight ways of locating the initial, apical, final and basical elements of a simple matrix when these lie at the four corners, and the apex and base are at opposite corners; and we have eight standard notations for the matrix, one corresponding to each of the eight allocations. They are obtained by supposing the corner elements to carry their names with them both when the order of arrangement of the horizontal or vertical rows is reversed, and when horizontal and vertical rows are interchanged. We can regard (A) and (B) together as forming a complete set of simple matrices correlated by the operations of

> reversing the order of arrangement of the horizontal rows,
> reversing the order of arrangement of the vertical rows,
> replacing a matrix by its conjugate.

A simple matrix of assigned type can be regarded as being built up with

(*i*) successive horizontal rows starting from the base,
(*ii*) successive vertical rows starting from the base,
(*iii*) successive diagonal lines starting from the base.

The standard notations will be helpful in considering these three aspects of the matrix simultaneously; and they will also be used to define special kinds of simple matrices in which the type is a material feature, and cannot be chosen arbitrarily.

NOTE 1. *The conjugate of a standard general simple matrix.*

The conjugates of the four direct standard general simple matrices $X=[x]_m^n$ of the respective types (*a*) are the four inverted standard general simple matrices $X'=\underset{\llcorner\lrcorner\, n}{\overset{\ulcorner\urcorner\, m}{x}}$ of the respective types

$$\theta_2=\{\pi', \pi'\}, \quad \theta_1=\{\pi, \pi\}, \quad \theta_3=\{\pi', \pi\}, \quad \theta_4=\{\pi, \pi'\}. \quad \ldots \quad (a')$$

Types represented by correspondingly situated symbolic commutants in the two sets (*a*) and (*a'*) are called *mutually conjugate types*.

NOTE 2. *Transition from one standard form to another.*

Let i, I and j, J be two pairs of positive integers such that

$$i+\mathrm{I}=m+1,\ (i=1,2,\dots m)\ ;\qquad j+\mathrm{J}=n+1,\ (j=1,2,\dots n)\ ;$$

and let $\mathrm{X}=[x]^{n}_{m}$, $\mathrm{Y}=[y]^{n}_{m}$ or $\mathrm{X}=\underset{\sqcup m}{\overset{\sqcap n}{x}}$, $\mathrm{Y}=\underset{\sqcup m}{\overset{\sqcap n}{y}}$

be two standard forms, both direct or both inverted, of the same matrix X having the non-ruled type θ. Then according as the type of Y is

$$\theta,\quad j\theta j,\quad j\theta,\quad \theta j,$$

we have $x_{ij}=y_{ij}$, $x_{ij}=y_{\mathrm{IJ}}$, $x_{ij}=y_{\mathrm{I}j}$, $x_{ij}=y_{i\mathrm{J}}$.

Again let $\mathrm{X}=[x]^{n}_{m}$, $\mathrm{Y}=\underset{\sqcup m}{\overset{\sqcap n}{y}}$ or $\mathrm{X}=\underset{\sqcup m}{\overset{\sqcap n}{x}}$, $\mathrm{Y}=[y]^{n}_{m}$

be two standard forms, one direct and the other inverted, of the same matrix X having the non-ruled type θ. Then according as the type of Y is

$$\theta,\quad j\theta j,\quad j\theta,\quad \theta j,$$

we have $x_{ij}=y_{\mathrm{JI}}$, $x_{ij}=y_{ji}$, $x_{ij}=y_{\mathrm{J}i}$, $x_{ij}=y_{j\mathrm{I}}$.

These equations provide us with substitutions by which a simple matrix expressed in any one standard form can be expressed in any other standard form. The most useful substitution is $x_{ij}=y_{\mathrm{JI}}$ which converts a simple matrix expressed in a direct standard form into the inverted standard form of the same type.

NOTE 3. *The natural types.*

The *natural types* are $\{\pi,\pi\}$ when a direct notation is used, and $\{\pi',\pi'\}$ when an inverted notation is used. A simple matrix $[x]^{n}_{m}$ to which no type is assigned is ordinarily assumed to have the type $\{\pi,\pi\}$, and to be expressed in the corresponding standard direct form; and a simple matrix $\underset{\sqcup n}{\overset{\sqcap m}{x}}$ to which no type is assigned is ordinarily assumed to have the type $\{\pi',\pi'\}$, and to be expressed in the corresponding standard form. Then $[x]^{n}_{m}$ and $\underset{\sqcup n}{\overset{\sqcap m}{x}}$ are mutually conjugate matrices having mutually conjugate non-ruled types.

2. *Parametric elements ; horizontal and vertical parametric rows ; major-parametric rows.*

In dealing with a simple matrix the use of the term *parametric* always indicates that the matrix is regarded as having an assigned type, and always has reference to the standard general simple matrix (direct or inverted) of the same class and type of which the given matrix is a particularisation. Moreover it serves to indicate an order of counting depending on the type, and to render terms applicable to a simple matrix of natural type also applicable to any simple matrix having an assigned type.

If $X=[x]^{n}_{m}$ or $X=\overset{\ulcorner\urcorner^{m}}{\underset{\llcorner\lrcorner_{n}}{x}}$ is a given simple matrix having an assigned non-ruled type, then by the *parametric element* x_{ij} of X we mean that element which would be denoted by x_{ij} in the corresponding standard general simple matrix, direct or inverted, of the same class and type. In each of the eight matrices (A) and (B), whether general or particularised, the *pth horizontal parametric row* is the *pth* horizontal row counting from the horizontal row through the initial parametric element x_{11}; the *qth vertical parametric row* is the *qth* vertical row counting from the vertical row through x_{11}; the *qth parametric element* of the *pth* horizontal parametric row is the element common to that row and the *qth* vertical parametric row; the *pth parametric element* of the *qth* vertical parametric row is the element common to that row and the *pth* horizontal parametric row. When the direct notation is used, the parametric element x_{ij} is common to the *ith* horizontal and *jth* vertical parametric rows; when the inverted notation is used, it is common to the *jth* horizontal and *ith* vertical parametric rows.

When we wish to emphasise or make clear the fact that the term parametric is being used in the way just described with reference to a standard general simple matrix, we will replace it by the term *major-parametric.* When a simple matrix of a special kind is regarded as a particularisation of a most general simple matrix of that special kind, it will be natural to use the term parametric in other ways.

3. *Simple matrices having assigned ruled types.*

The 16 ruled types, which include the 4 non-ruled types are represented by the 16 symbolic commutants

$$\theta_{11}=\{\pi,\pi\},\quad \theta_{12}=\{-\pi,-\pi\},\quad \theta_{13}=\{-\pi,\pi\},\quad \theta_{14}=\{\pi,-\pi\}\ ;\quad (a_1)$$
$$\theta_{21}=\{\pi',\pi'\},\quad \theta_{22}=\{-\pi',-\pi'\},\quad \theta_{23}=\{-\pi',\pi'\}\quad \theta_{24}=\{\pi',-\pi'\}\ ;\quad (a_2)$$
$$\theta_{31}=\{\pi',\pi\},\quad \theta_{32}=\{-\pi',-\pi\},\quad \theta_{33}=\{-\pi',\pi\},\quad \theta_{34}=\{\pi',-\pi\}\ ;\quad (a_3)$$
$$\theta_{41}=\{\pi,\pi'\},\quad \theta_{42}=\{-\pi,-\pi'\},\quad \theta_{13}=\{-\pi,\pi'\},\quad \theta_{44}=\{\pi,-\pi'\}\ ;\quad (a_4)$$

where $\theta_{11}=\theta_1,\quad \theta_{21}=\theta_2,\quad \theta_{33}=\theta_3,\quad \theta_{41}=\theta_4.$

In each of these four sets the first type is the *non-ruled equivalent* of all the four types; and a simple matrix to which any one of those four types is ascribed will be said to have the first of them as its *non-ruled equivalent type.* If we suppose that

is either 1 or -1, μ is either 1 or -1, the four sets of symbolic commutants can be represented concisely by the four symbols

$$\{\lambda\pi,\mu\pi\},\quad \{\lambda\pi',\mu\pi'\},\quad \{\lambda\pi',\mu\pi\},\quad \{\lambda\pi,\mu\pi'\} \quad \ldots \quad (b)$$

from which they can be derived respectively by putting in succession

$$[\lambda,\mu]=[1,1],\quad [\lambda,\mu]=[-1,-1],\quad [\lambda,\mu]=[-1,1],\quad [\lambda,\mu]=[1,-1]. \quad (1)$$

By the four direct standard general simple matrices $X=[x]_m^n$ of the class M $\binom{n}{m}$ and of the four types.

$$\theta_{u1},\quad \theta_{u2},\quad \theta_{u3},\quad \theta_{u1} \qquad (u=1, 2, 3, 4), \quad \ldots \quad (2)$$

we mean the matrices derived from the first of them (which has been defined in sub-article 1) by multiplying all elements of the

1st, 2nd, 3rd,...*ith*,...major-parametric horizontal rows by

$$1,\lambda,\lambda^2, \ldots \lambda^{i-1}, \ldots;$$

1st, 2nd, 3rd, ...*jth*,...major-parametric vertical rows by

$$1,\mu,\mu^2,\ldots\mu^{j-1},\ldots;$$

where λ and μ receive in succession the four sets of values given by (1), the initial parametric element x_{11} always remaining unaltered. The positions of the initial parametric elements, apex, final parametric element, base are to be the same in all four matrices as in the first, and the horizontal and vertical parametric rows of all four matrices are to be defined in exactly the same ways as those of the first matrix. For each of the four values of u, the four standard simple matrices x of the respective types (2) form a complete set of simple matrices correlated by the operations of :

> changing the signs of all elements in every alternate major-parametric horizontal row, starting with the second such row; .. (3)
>
> changing the signs of all elements in every alternate major-parametric vertical row, starting with the second such row. .. (4)

Again for each of the four values of v the four standard simple matrices X of the respective type

$$\theta_{1v},\quad \theta_{2v},\quad \theta_{3v},\quad \theta_{1v}, \qquad (v=1, 2, 3, 4), \quad \ldots \quad (2')$$

which are thus defined form a complete set of simple matrices correlated by the operation of :

> reversing the order of arrangement of the vertical rows ; .. (3′)
>
> reversing the order of arrangement of the horizontal rows ; .. (4′)

i.e. correlated by simple reversals ; and the 16 standard simple matrices X which are thus defined form a complete set of simple matrices correlated by the operations (3), (4) and (3′), (4′). These 16 matrices are the 16 *direct standard general simple matrices* $X=[x]^{n}_{m}$ of the class $M\binom{n}{m}$ and of the 16 ruled types which have been assigned to them. When we ascribe the type θ, non-ruled or ruled, to a simple matrix $[x]^{n}_{m}$ of the class $M\binom{n}{m}$, and say that it is expressed in the standard direct form appropriate to the type θ, we regard it as a particularisation of the direct standard general simple matrix of the same class and of the type θ, the same notations being used for its elements.

The 16 inverted standard general matrices $X=\overset{\sqcap\, n}{\underset{\sqcup\, m}{x}}$ of the 16 ruled types are derived in exactly the same way from the 4 inverted standard general simple matrices $X=\overset{\sqcap\, m}{\underset{\sqcup\, n}{x}}$ of the 4 non-ruled types defined in sub-article 1; and the same remarks can be made respecting them. They form a complete set of simple matrices correlated by the operations (3), (4) and (3′), (4′). When we assign the type θ, non-ruled or ruled, to a simple matrix $\overset{\sqcap\, m}{\underset{\sqcup\, n}{x}}$ of the class $M\binom{m}{n}$, and say that it is expressed in the standard inverted form appropriate to the type θ, we regard it as a particularisation of the inverted standard general simple matrix of the same class and of the type θ, the same notations being used for its elements.

We have now defined 32 standard general simple matrices in every class, *viz.* 16 direct of the 16 ruled types, and 16 inverted of the 16 ruled types; and we have 32 standard notations for the elements of any given simple matrix. We can however confine ourselves to the use of the direct notations; for we can always pass from the inverted notations to the direct notations. In dealing with special kinds of simple matrices, the inverted notations are often more convenient; but in the present paper we shall generlly use the direct notations. The ruled types, as distinguished from the non-ruled types, are only required when we are concerned with ruled simple matrices, *i.e.*, simple matrices in which the elements of each parametric diagonal line are either all equal or differ only in having signs which are alternately positive and negative.

If $X=[x]^{n}_{m}$ is a direct standard general simple matrix of any one of the types (b), the element common to its i th horizontal and j th vertical major-parametric rows is

$$\pm x_{ij} = \lambda^{i-1}\mu^{j-1}.x_{ij}, \qquad (\lambda=\pm 1,\ \mu=\pm 1); \qquad \dots \quad (5)$$

where x_{ij} is an arbitrary parameter which is the same for all the 16 types, and which will be called the parameter of that element. The same remarks apply when $X=\overset{\ulcorner\ m}{\underset{\llcorner\ n}{x}}$ is an inverted standard general simple matrix of any one of the types (b), except that we must replace (5) by

$$\pm x_{ji} = \lambda^{i-1}\mu^{j-1}.x_{ji}; \qquad (\lambda=\pm 1,\ \mu\pm 1). \qquad (5')$$

NOTE 4. *The conjugate of a standard general simple matrix.*

The conjugates of the direct standard general simple matrices $X=[x]_m^n$ of the class $M\binom{n}{m}$ and of the respective type

$$\{\lambda\pi, \mu\pi\}, \qquad \{\lambda\pi', \mu\pi'\}, \qquad \{\lambda\pi', \mu\pi\}, \qquad \{\lambda\pi, \mu\pi'\} \qquad \dots \quad (b)$$

are the inverted standard general simple matrices $X'=\overset{\ulcorner\ m}{\underset{\llcorner\ n}{x}}$ of the class $M\binom{m}{n}$ and of the respective types

$$\{\mu\pi', \lambda\pi'\}, \qquad \{\mu\pi, \lambda\pi\}, \qquad \{\mu\pi', \lambda\pi\}, \qquad \{\mu\pi, \lambda\pi'\}. \qquad \dots \quad (b')$$

Types represented by correspondingly situated symbolic commutants in the two sets (b) and (b') are *mutually conjugate types.*

NOTE 5. *Transition from the direct to the inverted notation.*

Let θ be any one of the types (b). Then if $X=[x]_m^n$ is a simple matrix expressed in the standard direct form appropriate to the type θ and if $Y=\overset{\ulcorner\ n}{\underset{\llcorner\ m}{y}}$ is the same matrix expressed in the standard inverted form appropriate to the same type θ, we have

$$\lambda^{i-1}\mu^{j-1}.x_{ij} = \lambda^{I-1}\mu^{J-1}.y_{JI}, \quad \text{or} \quad x_{ij} = \lambda^{m+1}\mu^{n+1}.y_{JI},$$

where $\qquad i+I=m+1, \qquad J+J=n+1.$

These equations give us substitution by which we can pass form the direct to the inverted form or from the inverted to the direct form without change of type.

4. *Commutantal products of simple matrices ; commutantal domains.*

Let

$$ABC\dots K\,L = X \qquad \dots \quad (6)$$

be a commutantal product of given simple matrices A,B,C,...K,L each of which has an assigned type ; let that type be assigned to X which renders the equation (6) commutantal, *i.e.* let the symbolic commutant which represents the type of X be the product of the symbolic commutants which represent the types of A, B, C,...K, L ; and let A, B, C,...K, L, X be expressed in the standard direct forms appropriate to the types assigned to them. Then the expressions for the parameters of X in terms of the parameters of A, B, C,...K, L are independent of the type of the product; and in finding them we can always replace (6) by any correlated commutantal product

$$A'B'C'\ldots K'L'=X' \qquad \ldots \quad (6')$$

We asually chose (6′) so that A′, B′, C′,...K′,L′ all have the ordinary type $\{\pi, \pi\}$.

The same remarks are applicable when A, B, C,...K, L are all expressed in the standard inverted forms appropriate to their types; and we may then chose (6′) so that A′, B′, C′,...K′, L′ all have the ordinary type $\{\pi', \pi'\}$.

Thus in evaluating the product matrix in any commutantal product of t factor matrices, we are simultaneously evaluating the product matrices in all the 4^{t+1} correlated commutantal products.

All simple matrices of a kind S to which commutantal types can be assigned from a *commutantal domain* when in every commutantal product such as (6) in which A, B, C,...K, L are simple matrices of the kind S having assigned types the product matrix X is always a simple matrix of the same kind S and of the type which renders the equation (6) commutantal.

2. The parametric diagonal lines of a simple matrix.

Let $X=[x]_m^n$ or $X=[x]_n^m$ be any simple matrix which has an assigned type and is expressed in the standard direct or inverted form appropriate to that type. The parametric element $\pm x_{i,j}$ or its parameter $x_{i,j}$ will be said to have

$$\text{difference-weight } j-i, \qquad \text{total-weight } i+j.$$

The matrix X contains two sets of diagonal lines, *viz.*

(1) $m+n-1$ mutually parallel diagonal lines each having the property that all the elements $\pm x_{i,j}$ lying on it have the same difference-weight $j-i$, the difference-weights of the successive lines starting from the base being

$$1-m, \quad 2-m, \ldots \quad n-2, \quad n-1;$$

(2) $m+n-1$ mutually parallel diagonal lines each having the property that all the elements $\pm x_{i,j}$ lying on it have the same

total weight $i+j$, the total weights of the successive lines starting from the initial parametric element being

$$2,\ 3,\ \ldots m+n-1,\ m+n.$$

We will lay the greater stress on the diagonal lines of constant difference-weight, and call these the *parametric diagonal lines.* They will ordinarily be counted from the base to the apex. The first or initial parametric element of any such line is that element $\pm x_{ij}$ lying on it for which i and j have the smallest values.

In any simple matrix those of the diagonal lines which slope downwards towards the right, or are parallel to the leading diagonal, are the *ante-diagonal lines*; those of them which slope downwards towards the left are the *counter-diagonal lines.* The diagonal lines drawn through the matrix from the four corners are the *diagonals* of the matrix, the diagonal drawn through the matrix from the top left-hand corner being the *leading diagonal.* When the matrix is not square, there are four diagonals, two of them being *ante-diagonals* and two being *counter-diagonals*; when the matrix is square, there are only two diagonals, *viz.* one ante-diagonal and one counter-diagonal. The leading diagonal is always an ante-diagonal. The meanings of these terms are independent of the type of the matrix and of the form in which it is expressed.

The diagonal lines of constant total weight drawn through the matrix from the apex and base (which become one when the matrix is square) will be called respectively the *apical diagonal* (or *median line*) and the *basical diagonal* of the matrix. The parametric diagonal lines (of constant difference-weight) drawn through the matrix from the initial and final parametric elements are the *parametric diagonals* (there being only one parametric diagonal when the matrix is square). We will distinguish between the two parametric diagonals by calling one of them the pre-major and the other the post-major diagonal. The *pre-major diagonal* is drawn through the matrix from the other corner lying in the same horizontal line as the apex (or the same vertical line as the base); the *post-major diagonal* is drawn through the matrix from the other corner lying in the same vertical line as the apex (or the same horizontal line as the base); the *non-major diagonal* is that one of the two parametric diagonals which lies nearer to the apex (or more remote from the base). The parametric diagonal lines extending from the pre-major diagonal to the apex, from the post-major diagonal to the apex, and from the non-major diagonal to the apex will be called respectively the *pre-major diagonal lines*, the *post-major diagonal lines*, and the *non-major diagonal lines.* The pre-major or post-major or non-major diagonal is the first pre-major or first post-major or first non-major diagonal line. The meanings of all these terms depend on and are fixed by the type; when the type or only its non-ruled equivalent is given, so that the positions of the apex and base are fixed, the meanings are independent of the form in which the matrix is expressed. The words 'pre-major' and 'post-major' have however been coined (See Exs. *i* and *ii*) with special reference to the direct standard forms.

In a simple matrix expressed in the standard form, direct or inverted, appropriate to one of the 8 types $\{\pm\pi, \pm\pi\}$, $\{\pm\pi', \pm\pi'\}$ the parametric diagonal lines are the ante-diagonal lines; and these types are therefore called the *ante-types.* In a simple matrix expressed in the standard from, direct or inverted, appropriate to one of the 8 types $\{\pm\pi', \pm\pi\}$, $\{\pi, \pm\pi'\}$ the parametric diagonal lines are the counter-diagonal lines; and these types are therefore called the *counter-types.* The conjugate of an ante-type is always an ante-type; and the conjugate of a counter-type is always a counter-type.

Ex. i If $\pm x_{ij}$ is a parametric element of a simple matrix $X=[x]_m^n$ or $X=\overset{\sqcap m}{\underset{\sqcup n}{x}}$ expressed in the standard direct or inverted form appropriate to its type, the possible values of i and j, the possible values of the difference-weight $k=j-i$, and the possible values of i and j when k is given are those consistent with the respective sets of conditions

$$i \nless 1,\ i \ngtr m,\ j \nless 1,\ j \ngtr n\ ; \qquad \dots \quad (1)$$

$$k \nless 1-m,\ k \ngtr n-1\ ; \qquad \dots \quad (1')$$

$$i \nless 1,\ i \nless 1-k,\ i \ngtr n-k,\ i \ngtr n\ ;\ j \nless 1,\ j \nless 1+k,\ j \ngtr m+k,\ j \ngtr n\ ; \qquad \dots \quad (1'')$$

i and j being horizontal and vertical indicators in $[x]_m^n$, and vertical and horizontal indicators in $\overset{\sqcap m}{\underset{\sqcup n}{x}}$ The two sets of inequalities (1″) show the horizontal and vertical or the vertical and horizontal parametric rows in which the parametric elements of difference-weight k lie.

The pre-major and post-major diagonals pass respectively through

the initial element x_{11} and the final element $\pm x_{mn}$ in $[x]_m^n$;

the final element $\pm x_{mn}$ and the initial element x_{11} in $\overset{\sqcap m}{\underset{\sqcup n}{x}}$.

Ex. ii If $\pm y_{ji}$ is a parametric element of simple matrix $Y=\overset{\sqcap n}{\underset{\sqcup m}{y}}$ or $Y=[y]_n^m$ expressed in the standard inverted or direct standard form appropriate to its type, the possible values of the difference-weight $k=i-j$, and the possible values of j and i when k is given are those consistent with the respective sets of conditions

$$j \nless 1,\ j \ngtr n,\ i \nless 1,\ i \ngtr m\ ; \qquad (2)$$

$$k \nless 1-n,\ k \ngtr m-1\ , \qquad (2')$$

$$j \nless 1,\ j \nless 1-k,\ j \ngtr m-k,\ j \ngtr n\ ,\ i \nless 1,\ i \nless 1+k,\ i \ngtr n+k,\ i \ngtr m\ ; \qquad (2'')$$

j and i being vertical and horizontal indicators in $\overline{\underline{y}}{}^{n}_{m}$, and horizontal and vertical indicators in $[y]^{n}_{m}$.

The pre-major and post-major diagonals pass respectively through :

the final element $\pm y_{nm}$ and the initial element y_{11} in $\overline{\underline{y}}{}^{n}_{m}$;

the initial element y_{11} and the final element $\pm y_{nm}$ in $[y]^{m}_{n}$.

3. The nullities and the parametric rank of a simple matrix of given type.

1. *The basical nullity and the parametric (or apical) rank.*

Let $X=[x]^{n}_{m}$ or $X=\overline{\underline{x}}{}^{m}_{n}$ be any given simple matrix of the class $M\binom{n}{m}$ or $M\binom{m}{n}$ which has an assigned type and is expressed in the standard direct or inverted form appropriate to the type.

We will define the *basical nullity* $\bar{\xi}$ of X to be the exact number of consecutive parametric diagonal lines counting from the base which contain only zero elements, the possible values of $\bar{\xi}$ being given by

$$\bar{\xi} \nless 0,\ \bar{\xi} \ngtr m+n-1.$$

If X is a zero matrix, we have $\bar{\xi}=m+n-1$. If X is not a zero matrix, so that $\bar{\xi}<m+n-1$, the first $\bar{\xi}$ parametric diagonal lines of X counting from the base contain only zero elements, but the next parametric diagonal line, which will be called the *first effective parametric diagonal line*, contains at least one non-zero element. In this latter case all elements of X having difference-weights not greater than $\bar{\xi}-m$ are equal to 0, the least difference-weight of a non-zero element being $\bar{\xi}-m+1$; and the parametric diagonal lines extending from the first effective parametric diagonal line to the apex will be called the *effective parametric diagonal lines*, their difference-weights being

$$\bar{\xi}-m+1,\ \bar{\xi}-m+2, \ldots n-2,\ n-1.$$

The last definition remains valid even when some or all of the parametric diagonal lines coming after the first effective parametric diagonal line contain only zero elements.

In place of the basical nullity $\bar{\xi}$ it is generally more convenient to use the *parametric* (or *apical*) rank ξ, which we define to be the total number of the effective parametric diagonal lines. It is the positive integer ξ determined by the equation

$$\xi+\bar{\xi}=m+n-1, \qquad \dots \quad (1)$$

the possible values of which are those consistent with the conditions

$$\xi \nless 0, \quad \xi \ngtr m+n-1.$$

Because of the relation (1) we can always put

$$\xi=n-\xi_0, \quad \bar{\xi}=m-1+\xi_0, \qquad \dots \quad (2)$$

where ξ_0 is a uniquely determinate integer which is subject to the conditions

$$\xi_0 \nless 1-m, \quad \xi_0 \ngtr n.$$

When $\xi_0=n$, X is a zero matrix; if $\xi_0<n$, then ξ_0 is the least difference-weight of a non-zero element of X, *i.e.* the difference-weight of the first effective parametric diagonal line. In all cases we will call ξ_0, the *least effective difference-weight* of X, interpreting it to be n when X is a zero matrix.

The excess of the total number of effective parametric diagonal lines over the total number of pre-major parametric diagonal lines (or vertical rows) will be called the *pre-major parametric excess.* It is an integer which may be either positive or negative, and when its sign is changed, it becomes the pre-major parametric *defect.* The pre-major parametric excess is

$$\xi-n \text{ when } X=[x]_m^n, \quad \xi-m \text{ when } X=\overset{\sqcap^m}{\underset{\sqcup_n}{x}}.$$

The excess of the total number of effective parametric diagonal lines over the total number of post-major parametric diagonal lines (or horizontal rows) will be called the *post-major parametric excess.* It is an integer which may be either positive or negative; and when its sign is changed, it becomes the post-major parametric *defect.* The post-major parametric excess is

$$\xi-m \text{ when } X=[x]_m^n, \quad \xi-n \text{ when } X=\overset{\sqcap^m}{\underset{\sqcup_n}{x}}.$$

2. *The horizontal and vertical nullities.*

The *horizontal nullity* of a simple matrix having an assigned type is the total number of consecutive rows, counting from the horizontal row through the base, which contain only zero elements; consequently it is the total number of

final horizontal or initial horizontal

zero major-parametric rows according as the direct or inverted standard notation appropriate to the type is used.

The *vertical nullity* of a simple matrix having an assigned type is the total number of consecutive vertical rows, counting from the vertical row through the base, which contain only zero elements; consequently it is the total number of

initial vertical or final vertical

zero major-parametric rows according as the direct or inverted standard notation appropriate to the type is used.

When the direct forms are regarded as fundamental, we will denote the horizontal and vertical nullities of a simple matrix X by ξ_1 and ξ_2 or by ξ_2 and ξ_1 according as X is expressed in the direct or inverted standard form appropriate to its type.

First let $X=[x]_m^n$ be a simple matrix of the class $M\binom{n}{m}$ which is expressed in the standard direct form appropriate to its type, and which has

parametric rank ξ, horizontal nullity ξ_1, vertical nullity ξ_2,

where ξ, ξ_1, ξ_2 are essentially positive integers. If X is not a zero matrix, we have

$$\xi>0, \quad \xi_1<m, \quad \xi_2<n,$$

and we can express it in the 1st, 2nd, 3rd, 4th of the forms

$$\begin{bmatrix} 0, & x' \\ 0, & 0 \end{bmatrix}_{m-\xi_1,\ \xi_1}^{\xi_2,\ n-\xi_2}, \quad \begin{bmatrix} 0, & 0 \\ x', & 0 \end{bmatrix}_{\xi_1,\ m-\xi_1}^{n-\xi_2,\ \xi_2}, \quad \begin{bmatrix} 0, & 0 \\ x', & 0 \end{bmatrix}_{\xi_1,\ m-\xi_1}^{\xi_2,\ n-\xi_2},$$

$$\begin{bmatrix} x', & 0 \\ 0, & 0 \end{bmatrix}_{m-\xi_1,\ \xi_1}^{n-\xi_2,\ \xi_2} \quad \ldots \quad (A)$$

according as its type or the non-ruled equivalent of its type is represented by the 1st, 2nd, 3rd, 4th of the symbolic commutants

$$\theta_1=\{\pi, \pi\},\ \theta_2=\{\pi', \pi'\},\ \theta_3=\{\pi', \pi\},\ \theta_4=\{\pi, \pi'\} \quad \ldots \quad (a)$$

In each case the constituent $X_0 = [x']^{n-\xi_2}_{m-\xi_1}$ of X is a simple matrix having the same non-ruled equivalent type as X, and having

parametric rank ξ, horizontal nullity 0, vertical nullity 0 ;

and we will call X_0 the *least effective constituent* of X. The horizontal and vertical rows of X which occur in X_0 will be called the *effective* horizontal and vertical *parametric rows* of X, the $\substack{\text{last horizontal}\\ \text{first vertical}}$ parametric row of X_0 being (when extended) the $\substack{\text{last effective horizontal}\\ \text{first effective vertical}}$ parametric row of X, *i.e.* the $\substack{\text{last horizontal}\\ \text{first vertical}}$ parametric row of X which contains a non-zero element. According to previous definitions the effective parametric diagonal lines of X_0 are (when extended) the effective parametric diagonal lines of X.

Next let $X = [x]^{m}_{n}$ be a simple matrix of the class $M\binom{m}{n}$ which is expressed in the standard inverted forms appropriate to its type and has

parametric rank ξ, horizontal nullity ξ_2, vertical nullity ξ_1,

where ξ, ξ_1, ξ_2 are essentially positive integers. If X is not a zero matrix, we have

$$\xi > 0, \quad \xi_1 < m, \quad \xi_2 < n,$$

and we can express it in the 1st, 2nd, 3rd, 4th of the forms

$$\begin{bmatrix}0, & x'\\ 0, & 0\end{bmatrix}^{\xi_1,\ m-\xi_1}_{n-\xi_2,\ \xi_1}, \quad \begin{bmatrix}0, & 0\\ x', & 0\end{bmatrix}^{m-\xi_1,\ \xi_1}_{\xi_2,\ n-\xi_2}, \quad \begin{bmatrix}0, & 0\\ 0, & x'\end{bmatrix}^{\xi_1,\ m-\xi_1}_{\xi_2,\ n-\xi_2}, \quad \begin{bmatrix}x', & 0\\ 0, & 0\end{bmatrix}^{m-\xi_1,\ \xi_2}_{n-\xi_2,\ \xi_2} \qquad \ldots \quad (B)$$

according as its type or the non-ruled equivalent of its type is represented by the 1st, 2nd, 3rd, 4th of the symbolic commutants (a).

In each case the constituent $X_0 = [x']^{m-\xi_1}_{n-\xi_2}$ of X is a simple matrix having the same non-ruled equivalent type as X, and having

parametric rank ξ, horizontal nullity 0, vertical nullity 0 ;

and X_0 is the least effective constituent of X. The horizontal and vertical rows of X which occur in X_0 are the *effective* horizontal and vertical *parametric* rows of X, the $\begin{array}{l}\text{first horizontal}\\ \text{last vertical}\end{array}$ parametric row of X_0 being (when extended) the $\begin{array}{l}\text{first effective horizontal}\\ \text{last effective vertical}\end{array}$ parametric row of X, *i.e.* the $\begin{array}{l}\text{first horizontal}\\ \text{last vertical}\end{array}$ parametric row of X which contains a non-zero element. The effective parametric diagonal lines of X_0 are (when extended) the effective parametric diagonal lines of X. We could of course regard the 1st, 2nd, 3rd, 4th of the matrices (B) and their least effective constituents as the conjugates of the 2nd, 1st, 3rd, 4th of the matrices (A) and their least effective constituents.

When $X=[x]_m^n$ has the natural type $\{\pi,\pi\}$ or $X={\ulcorner x \lrcorner}^{m}_{n}$ has the natural type $\{\pi', \pi'\}$, the horizontal nullity is the total number of consecutive final horizontal rows of 0's, and the vertical nullity is the total number of consecutive initial vertical rows of 0's.

Whenever a simple matrix X which is expressed in the standard direct or inverted form appropriate to its type has that one of the forms (A) or (B) which corresponds to its type, the non-zero constituent occupying the corner at the apex, we will call X_0 the *effective constituent* of X. But ξ_1 and ξ_2 or ξ_2 and ξ_1 are the horizontal and vertical nullities only when X_0 is the *least* effective constituent, *i.e.* when X_0 has zero horizontal and vertical nullities, so that ξ_1 and ξ_2 or ξ_2 and ξ_1 have their greatest possible values.

In the following examples X in defined either as in (A) or as in (B), so that in both cases the relations (1) and (2) hold good.

Ex. i. When X is a zero matrix, the least effective constituent X_0 is non-existent and we have

$$\xi=0, \quad \xi_1=m, \quad \xi_2=n$$

Ex. ii. When X is a non-zero matrix having parametric rank ξ and zero horizontal and vertical nullities, the possible values of ξ are those consistent with the conditions

$$\xi \nless m, \quad \xi \nless n, \quad \xi \ngtr m+n-1.$$

In fact the characteristic property of a non-zero simple matrix having zero horizontal and vertical nullities is that its parametric rank cannot be less than either of its two orders.

Ex. iii. When X is a non-zero simple matrix, the possible simultaneous values of the positive integers ξ, ξ_1, ξ_2 are those consistent with the conditions

$$\xi_1 \nless 0,\ \xi_2 \nless 0,\ \ \xi+\xi_1 \nless m,\ \ \xi+\xi_2 \nless n,\ \xi+\xi_1+\xi_2 \ngtr m+n-1\ ;$$

which include the conditions

$$\xi \nless 1,\ \xi \ngtr m+n=1\ ;\quad \xi_1 \nless 0,\ \xi_1 \ngtr m-1,\ \xi_2 \nless 0,\ \xi_2 \ngtr n-1$$

giving respectively the possible values of ξ, and the possible simultaneous values of ξ_1 and ξ_2.

This follows from Ez. iii and the forms (A) or (B) into which X can be put.

Ex. iv. Under the same circumstances the possible simultaneous values of ξ_0, ξ_1, ξ_2 are those consistent with the conditions.

$$\xi_1 \nless 0,\ \xi_2 \nless 0,\ \xi_0-\xi_1 \ngtr n-m,\ \xi_0-\xi_2 \ngtr 0,\ \xi_1+\xi_2=\xi_0 \ngtr m-1,$$

which include $\xi_0 \nless 1-m,\ \xi_0 \ngtr n-1$;

and the possible simultaneous values of $\bar{\xi}$, ξ_1, ξ_2 are those consistent with the conditions

$$\xi_1 \nless 0,\ \xi_2 \nless 0,\ \bar{\xi} \ngtr m+n-2,\ \bar{\xi}-\xi_1 \ngtr n-1,\ \bar{\xi}-\xi_2 \ngtr m-1,\ \xi_1+\xi_2-\bar{\xi} \ngtr 0,$$

which include $\bar{\xi} \nless 0,\quad \bar{\xi} \ngtr m+n-2.$

In place of the essentially positive integers ξ_1, ξ_2 it will often be more convenient to use the essentially positive integers

$$\xi_{10}=\xi+\xi_1-m,\quad \xi_{20}=\xi+\xi_2-n.$$

Then ξ_{10} is the post-major, ξ_{20} the pre-major excess of X_0

$$\text{when } X=[x]_m^n$$

and ξ_{10} is the pre-major, ξ_{20} the post-major excess of X_0

$$\text{when } X=\underset{\sqcup\,n}{\overset{\sqcap\,m}{x}}\ .$$

The three nullities of X (basical, horizontal, vertical) are known when the values of ξ, ξ_1, ξ_2 or the values of ξ, ξ_{10}, ξ_{20} are given.

Ex. v. When X is a zero matrix, we have

$$\xi=0,\quad \xi_{10}=0,\quad \xi_{20}=0.$$

Ex. vi. When X is a non-zero matrix, the possible simultaneous values of the positive integers ξ, ξ_{10}, ξ_{20} are those consistent with the conditions

$$\xi_{10} \nless 0,\ \xi_{10} \nless \xi-m,\ \xi_{20} \nless 0,\ \xi_{20} \nless \xi-n,\ \xi_{10}+\xi_{20} \ngtr \xi-1,$$

which include the conditions

$$\xi \nless 1,\ \xi \ngtr m+n-1\ ;\ \xi_{10} \nless 0,\ \xi_{10} \ngtr n-1,\ \xi_{20} \nless 0,\ \xi_{20} \ngtr m-1$$

giving respectively the possible values of ξ, and the possible simultaneous values of ξ_{10} and ξ_{20}.

NOTE.—*Transition from the direct to the inverted forms.*

When the inverted forms are regarded as fundamental, we will denote the horizontal and vertical nullities of a simple matrix Y by η_1 and η_2 or by η_2 and η_1 according as Y is expressed in the inverted or direct standard form appropriate to its type. Then if the typical parametric element of Y is y_{ij} having difference-weight $k=i-j$, we can convert all the formulae relating to $X=[x]^n_m$ or $X=\underset{\sqcup\, n}{\overset{\sqcap\, m}{x}}$ which have been obtained into the corresponding formulae relating to a simple matrix

$Y=\underset{\sqcup\, n}{\overset{\sqcap\, n}{y}}$ having

parametric rank η, horizontal nullity η_1, vertical nullity η_2

or $Y=[y]^m_n$ having

parametric rank η, horizontal nullity η_2, vertical nullity η_1

by replacing $m, n;\ i, j;\ \xi_1, \xi_2;\ \xi_{10}, \xi_{20};\ \xi, \bar{\xi}, \xi_0, k, X, X_0$

by $n, m;\ j, i;\ \eta_2, \eta_1;\ \eta_{20}, \eta_{10};\ \eta, \bar{\eta}, \eta_0, k, Y, Y_0$

We then have $\eta_{10}=\eta+\eta_1-m,\quad \eta_{20}=\eta+\eta_2-n;$

where η_{10} is the post-major, η_{20} the pre-major excess of Y_0 when $Y=\underset{\sqcup\, m}{\overset{\sqcap\, n}{y}}$,

and η_{10} is the pre-major, η_{20} the post-major excess of Y_0 when $Y=[y]^m_n$.

3. *Simple slopes.*

Let parametric diagonal lines be counted from the base. Then a simple matrix of any given class which is expressed in the standard direct or inverted form appropriate to the type θ will be called :

a *simple pre-major slope* of that class and of the type θ when all parametric diagonal lines preceding the pre-major diagonal contain only zero elements,

or the effective parametric diagonal lines are all pre-major diagonal lines,

or the parametric rank does not exceed the number of vertical rows,

or the pre-major parametric excess is not greater than 0 ;

a *simple post-major slope* of that class and of the type θ when all parametric diagonal lines preceding the post-major diagonal contain only zero elements,

or the effective parametric diagonal lines are all post-major diagonal lines, or the parametric rank does not exceed the number of horizontal rows,

or the post-major parametric excess is not greater than 0 ;

a *simple non-major slope* of that class and of the type θ when all parametric diagonal lines preceding the non-major diagonal (the parametric diagonal lying nearer to the apex) contain only zero elements,

or the effective parametric diagonal lines are all non-major diagonal lines,

or the parametric rank does not exceed the smaller or effective order of the matrix,

or the pre-major and post-major parametric excesses are both not greater than 0.

Thus a *simple non-major slope* of a given class and type is a matrix which is both a simple pre-major slope and a simple post-major slope of that class and type. By a *simple major slope* of a given class and type will be meant a matrix which is either a simple pre-major slope or a simple post-major slope of that class and type.

The above definitions depend only on the class and the non-ruled equivalent of the type, *i.e.* on the class and the position assigned to the apex, being independent (when these are given) of the form in which the matrix is expressed ; but the most general form of a simple pre-major or post-major or non-major slope of a given class and type does depend on the position assigned to the apex, *i.e.* on the non-ruled equivalent of the type. Consequently in every class we have simple pre-major ond post-major and non-major slopes :

(*i*) of the type $\{\pi, \pi\}$ to which any one of the four types $\{\pm\pi, \pm\pi\}$ can be ascribed ;

(*ii*) of the type $\{\pi', \pi'\}$ to which any one of the four types $\{\pm\pi', \pm\pi'\}$ can be ascribed ;

(*iii*) of the type $\{\pi', \pi\}$ to which any one of the four types $\{\pm\pi', \pm\pi\}$ can be ascribed ;

(*iv*) of the type $\{\pi, \pi'\}$ to which any one of the four types $\{\pm\pi, \pm\pi'\}$ can be ascribed.

These will be called respectively simple pre-major and post-major and non-major slope of the *first, second, third, fourth* types. Those having the ante-types (*i*) and (*ii*) will be called simple pre-major and post-major and non-major *ante-slopes*, because in them the parametric diagonal lines are ante-diagonal lines. Those having the counter-types (*iii*) and (*iv*) will be called simple pre-major and post-major and non-major *counter-slopes*, because in them the parametric diagonal lines are counter-diagonal lines.

Ex. vii. The simple matrix $X=[x]_m^n$ to which the type θ is ascribed is :

a simple *pre-major slope* of type θ when $\xi \ngtr n$, or $\xi_0 \nless 0$;

a simple *post-major slope* of type θ when $\xi \ngtr m$, or $\xi_0 \nless n-m$;

a simple *non-major slope* of type θ when $\xi \ngtr m$, $\xi \ngtr n$; or $\xi_0 \nless 0$, $\xi_0 \nless n-m$.

The simple $X=\boxed{x}_{n}^{m}$ to which the type θ is ascribed is :

a simple *pre-major slope* of type θ when $\xi \ngtr m$, or $\xi_0 \nless n-m$;

a simple *post-major slope* of type θ when $\xi \ngtr n$, or $\xi_0 \nless 0$;

a simple *non-major slope* of type θ when $\xi \ngtr m$. $\xi \ngtr n$; or $\xi_0 \nless 0$, $\xi_0 \nless n-m$.

4. The product matrix in a commutantal product of general simple matrices.

We consider first the commutantal product

$$[a]_m^p\ [b]_p^q\ [c]_q^r\ [d]_q^n=[x]_m^n \quad \text{or} \quad ABCD=X \quad \dots \quad (A)$$

in which A, B, C, D are four direct general standard simple matrices of given commutantal types (non-ruled or ruled) ; we ascribe to X the type appropriate to the product, so that the symbolic commutant representing the type of X is the product of the successive symbolic commutants representing the types of A, B, C, D ; and we regard X as a simple matrix expressed in the standard direct form appropriate to its type. Then the element common to the i th horizontal and *jth* vertical parametric rows of X is $\pm x_{ij}$, where the sign depends on the type of X, and is known ; and X will be completely known when we know all its parameters such as x_{ij}. The matrices A, B, C, D have zero basical, zero horizontal, and zero vertical nullities, and the given parametric (or apical) ranks

$$\alpha=m+p-1,\ \beta=p+q-1,\ \gamma=q+r-1,\ \delta=q+n-1,$$

their parameters being all arbitrary and independent.

If x_{ij} is a parameter of X, the possible values of i and j, the possible values of the difference-weight $k=j-i$, and possible values of i and j when k is given are those consistent with the respective sets of conditions

$$i \nless 1,\ i \ngtr m,\ j \nless 1,\ j \ngtr n\ ; \qquad \ldots \quad (1)$$

$$k \nless 1-m,\ k \ngtr n-1\ ; \qquad \ldots \quad (1')$$

$$i \nless 1,\ i \nless 1-k,\ i \ngtr n-k,\ i \ngtr m\ ;\ j \nless 1,\ j \nless 1+k,\ j \ngtr m+k,\ j \ngtr n. \qquad \ldots \quad (1'')$$

The two sets of conditions (1″) are mutually equivalent, and show respectively the horizontal and vertical parametric rows of X in which the elements x_{ij} having a given possible difference-weight k lie; and each of them together with (1′) and the equation $k=j-i$ is equivalent to (1).

There are 4^5 possible types of the commutantal product (A); but in all of them the expression for any given parameter x_{ij} of X as a rational integral function of the parameters of A, B, C, D is the same, being independent of the type, and in obtaining it we may suppose A, B, C, D, X to all have the same natural type $\{\pi, \pi\}$. Accordingly (see § 51 of *Matrices and Determinoids, Vol. 1*), it is the expression given in the following theorem :

Theorem I. *Every parametric element of the product matrix X in the commutantal product (A) is given by the general formula*

$$x_{ij} = \Sigma a_{iu}\, b_{uv}\, c_{vw}\, d_{wj}, \quad (j-i=k), \qquad \ldots \quad (a)$$

where the possible values of i, j and k are given by (1), (1′), (1″), and where for a given pair of values of i and j there are pqr terms in the sum (a) obtained by giving to u, v, w the values

$$u=1, 2, \ldots p\ ; \quad v=1, 2, \ldots q\ ; \quad w=1, 2, \ldots r.$$

Consequently none of the elements of X vanish identically ; the basical, horizontal and vertical nullities of X are all equal to O ; and X has the parametric rank ξ given by

$$\xi = m+n-1.$$

The summation in (a) extends over all distinct sets of values of u, v, w which can be formed when u, v, w are elements of the respective sequences

$$[1, 2, \ldots p], \quad [1, 2, \ldots q], \quad [1, 2, \ldots r].$$

The second part of the theorem is a consequence of the fact that the parameters of A, B, C, D are all arbitrary and independent, there being no relations between them. The expression for x_{ij} is the same for all possible values of k; but the elements of X having a given difference-weight k lie only in the horizontal and vertical parametric rows of X given by the first and second of the two sets of conditions (1″). It is particularly important to observe that :

The expression (a) for a parameter x_{ij} *of X having a given difference-weight* $j-i=k$ *is isobaric of weight k in the difference-weights of the parameters of A, B, C, D; for we have*

$$(u-i)+(v-u)+(w-v)+(j-w)=j-i=k.$$

In every commutantal product of four simple matrices of the classes M $\binom{p}{m}$, M $\binom{q}{p}$, M $\binom{r}{q}$, M $\binom{n}{r}$ the factor matrices are particularisations of A, B, C, D in the commutantal product (A) of the same type, and the product matrix is a particularisation of X. But when A, B, C, D are particularised by assigning particular values to some of their parameters or by introducing relations between the parameters, then in the expression for x_{ij}, all the pqr terms may vanish, in which case we certainly have $x_{ij}=0$, or some only of those terms may vanish, in which case the expression for x_{ij} can be simplified by the omission of the zero terms. The following articles deal with certain particularisations of the product (A) and the corresponding simplifications of the formula (a).

When A, B, C, D are particularised in any way, so that X also is particularised, let $\alpha_0, \beta_0, \gamma_0, \delta_0, \xi_0$ be the least effective difference-weight of A, B, C, D, X. Then if x_{ij} is any element of X having difference-weight $k=j-i$, and if there exists a non-zero term $a_{iu}\, b_{uv}\, c_{vw}\, d_{wj}$, in the expression ($a$) for x_{ij}, we must have

$$u-i \nless \alpha_0,\; v-u \nless \beta_0,\; w-v \nless \gamma_0,\; j-w \nless \delta_0,$$

and therefore $$k \nless \alpha_0+\beta_0+\gamma_0+\delta_0. \qquad \ldots \quad (2)$$

Consequently we have the following general theorem relating to all such particularisations :

Theorem II. *Whenever in any particularisation of the product (A) the product matrix X is not a zero matrix, its least effective difference-weight* ξ_0 *cannot be less than the sum of the least effective difference-weights* $\alpha_0, \beta_0, \gamma_0$ *of the factor matrices A, B, C, D; and every element of X whose difference-weight is less than this sum must be equal to 0.*

Many important conclusions can be drawn from this result. First, let A, B, C, D be all simple pre-major slopes, so that

$$\alpha_0 \nless 0,\; \beta_0 \nless 0,\; \gamma_0 \nless 0,\; \delta_0 \nless 0.$$

Then if X is not a zero matrix, we must have $\xi_o \not< 0$; and it follows that :

In a commutantal product of simple pre-major slopes, the product matrix is always a simple pre-major slope of the type appropriate to the product.

Next let A, B, C, D be all simple post-major slopes, so that

$$\alpha_o \not< p-m,\ \beta_o \not< q-p,\ \gamma_o \not< r-q,\ \delta_o \not< n-m.$$

Then if X is not a zero matrix, we must have $\xi_o \not< n-m$; and it follows that :

In a commutantal product of simple post-major slopes, the product matrix is always a simple post-major slope of the type appropriate to the product.

Lastly let A, B, C, D be all simple (non-major) slopes, so that each of them is both a simple pre-major slope and a simple post-major slope. Then if X is not a zero matrix, we must have $\xi_o \not< 0$, $\xi_o \not< n-m$; and it follows that :

In a commutantal product of simple (non-major) slopes, the product matrix is always a simple (non-major) slope of the type appropriate to the product.

These three results show that :

All simple pre-mojor slopes, all simple post-major slopes, and all simple non-major slopes form three commutantal domains, the third domain being included in each of the other two.

If when the product (A) is particularised in any way, the parametric ranks of A, B, C, D; X are $\alpha, \beta, \gamma, \delta, \xi$, we have

$$\alpha_o = p-\alpha,\ \beta_o = q-\beta,\ \gamma_o = r-\gamma,\ \delta_o = n-\delta,\ \xi_o = n-\xi,$$

and we can put

$$\alpha_o+\beta_o+\gamma_o+\delta_o = n-\rho, \qquad \dots \quad (3)$$

$$\rho = \alpha+\beta+\gamma+\delta-p-q-r. \qquad \dots \quad (4)$$

When all possible values are given to $\alpha, \beta, \gamma, \delta$, the possible values of the integer ρ are those consistent with the conditions

$$\rho \not< -(p+q+r),\ \rho \not> m+p+q+r+n-4.$$

When X is not a zero matrix, we see from (3) and Theorem II that we must have $\xi_0 \nless n-\rho$, and because we must also have $\xi_0 \ngtr n-1$, it follows that $\rho \nless 1$, or $\rho > 0$; *i.e.* we have the following theorem :

Theorem III. *If* $\rho \ngtr 0$ *in any particularisation of the product* (*A*), *then* X *is necessarily a zero matrix ; and whenever* X *is a non-zero matrix, we must have*

$$\rho > 0 ; \quad \text{and } \xi \nless 1, \quad \xi \ngtr \rho.$$

The integer ρ will occur in all the following articles. If we replace (A) by the commutantal product

$$\text{I A B C D J} = \text{X}$$

in which $\text{I} = [1]_m^m$ and $\text{J} = [1]_n^n$, it is (see Art. 7) the discriminator of the product I A B C D J. It will be seen in the following articles that $\rho > 0$ is only a necessary and not a sufficient condition for X being a non-zero matrix.

5. The product matrix in a commutantal product of most general simple matrices having zero horizontal and vertical nullities and given parametric ranks.

We will next consider the commutantal product

$$[a]_m^p \; [b]_p^q \; [c]_q^r \; [d]_r^n = [x]_m^n \quad \text{or} \quad \text{ABCD} = \text{X} \quad \dots \quad \text{(B)}$$

in which A, B, C, D are four most general simple matrices having zero horizontal and vertical nullities and given parametric ranks α, β, γ, δ, so that the possible values of α, β, γ, δ are those consistent with the conditions

$$\alpha \nless m, \; \alpha \nless p, \; \alpha \ngtr m+p-1 \; ; \qquad \beta \nless p, \; \beta \nless q, \; \beta \ngtr p+q-1 \; :$$

$$\gamma \nless q, \; \gamma \nless r, \; \gamma \ngtr q+r-1 \; ; \qquad \delta \nless r, \; \delta \nless n, \; \delta \ngtr q+n-1 \; ; \quad \dots \quad (1)$$

and as before we will put

$$\rho = \alpha + \beta + \gamma + \delta - p - r - q \qquad \dots \quad (2)$$

$$= m + (\alpha - m) + (\beta - p) + (\gamma - q) + (\delta - r) = (\alpha - p) + (\beta - q) + (\gamma - r) + (\delta - n) + n,$$

so that ρ is now a non-zero positive integer which satisfies the conditions

$$\rho \nless m, \; \rho \nless p, \; \rho \nless q, \; \rho \nless r, \; \rho \nless n, \; \rho \ngtr m+p+q+r+n-4.$$

The product (B) is derived from the product (A) by putting equal to 0 all elements (or parameters) of A, B, C, D having difference-weights less than α_o, β_o, γ_o, δ_o respectively, where

$$\alpha_o = p-\alpha,\ \beta_o = q-\beta,\ \gamma_o = r-\gamma,\ \delta_o = n-\delta, \quad \ldots \quad (3)$$

the other parameters of A, B, C, D remaining arbitrary and independent. It is assumed that A, B, C, D, X are expressed in the standard direct forms appropriate to their types, X having the type which renders the equation (B) commutantal.

When i and j are integers satisfying the conditions

$$i \nless 1,\ i \ngtr m,\ j \nless 1,\ j \ngtr n, \quad \ldots \quad (4)$$

the parameter x_{ij} of X is given by the formula (a) of Art. 4, and the expression (a) will contain a non-zero term $a_{iv}\, b_{uv}\, c_{vw}\, d_{wj}$ when and only when u, v, w, are integers satisfying the conditions

$$u \nless 1,\ u \ngtr p,;\ v \nless 1,\ v \ngtr q;\ w \nless 1,\ w \ngtr r; \quad \ldots \quad (4')$$

$$u-i \nless p-\alpha,\ v-u \nless q-\beta,\ w-v \nless r-\gamma,\ j-w \nless n-\delta \quad \ldots \quad (4'')$$

Again because the non-zero parameters of A, B, C, D are all arbitrary and independent, the expression (a) cannot vanish identically unless every one of its terms vanishes identically. Consequently x_{ij} will be a non-zero parameter of X when and only when there exist integers i, j, u, v, w, satisfying the conditions (4), (4′), (4″). Hence by eliminating u, v, w we see that when $\pm\, x_{ij}$ is a non-zero term of X the possible values of i and j, the possible values of the difference-weight

$$k = j-i, \quad \ldots \quad (5)$$

and the possible values of i and j when k is given are those consistent with the respective sets of conditions

$$i \nless 1,\ k \ngtr m,\ j \nless 1,\ j \ngtr n,\ j-i \nless n-\rho; \quad \ldots \quad (6)$$

$$k \nless 1-m,\ i \ngtr m-\rho,\ k \ngtr n-1;\ \text{or}\ k \nless \xi_o,\ k \ngtr n-1 \quad \ldots \quad (6')$$

$$i \nless 1,\ i \nless 1-k,\ i \ngtr n-k,\ i \ngtr m;\ j \nless 1,\ j \nless 1+k,\ j \ngtr m+k,\ j \ngtr n \quad \ldots \quad (6'')$$

where ρ is the integer (2), and where ξ_o is the greater of the two integers $1-m$, $n-\rho$, and is the least difference-weight of a non-zero element of X. We could arrive at the same conclusions by observing as in Art. 4 that $j-i \nless n-\rho$ is a necessary condition that the expression (a) shall contain a non-zero term, and then using the formula (b). By eliminating first j and then i from the conditions

(6) we see that there is a non-zero element in every horizontal and every vertical row of X. In this way we obtain the following two theorems :

Theorem I. *In the commutantal product (B) the product matrix X is a non-zero simple matrix having zero horizontal and vertical nullities and parametric rank ξ, where*

$\xi = n - \xi_0$ is the smaller of the two non-zero positive integers

$$\rho,\ m+n-1.$$

the smallest difference-weight of a non-zero element of X being ξ_0, where

$\xi_0 = n - \xi$ is the greater of the two negative integers $n-\rho$, $1-m$.

Theorem II. *The elements of the ξ effective parametric diagonal lines of X are all different from 0, i.e. none of them vanish identically, and their parameters are given by the general formula*

$$x_{ij} = \Sigma a_{iu}\, b_{uv}\, c_{vw}\, d_{wj}, \qquad (j-i=k) \qquad \ldots \quad (b)$$

where the possible values of i, j and k are given by (6), (6′), (6″), and where for a given pair of values of i and j the summation extends over all distinct sets of values of u, v, w which can be formed when u, v, w aae elements of the respective sequences

$$[1, 2, \ldots p], \quad [1, 2, \ldots q], \quad [1, 2, \ldots r]$$

which make every one of the four terms on the left positive in the identical equation

$$\{(u-i)-\alpha_0\} + \{(v-u)-\beta_0\} + \{(w-v)-\gamma_0\} + \{(j-w)-\delta_0\}$$
$$= k-(n-\rho). \quad \ldots \quad (b')$$

Here α_0, β_0, γ_0, δ_0 are the integers (3), which are the least difference-weights of non-zero elements of A, B, C, D.

Thus X has the form of a most general simple matrix having zero horizontal and vertical nullities and parametric rank ξ, in so far as none of its effective elements or parameters are equal to 0 ; but those effective parameters are rational integral functions of the effective parameters of A, B, C, D, and in saying that none of them are equal to 0 we mean that none of them vanish identically. When we particularise A, B, C, D by giving particular values to some of their effective parameters or by introducing relations between the effective parameters, some of the non-zero terms of X may be replaced by zero terms, but every zero element of X will necessarily remain a zero element in the particularisation of X. Thus the particularisation of X may have non-zero horizontal and vertical nullities, and its parametric rank may be less than (but can never be greater than) the integer ξ defined in Theorem I.

The number of terms in the expression on the left in (*b*) depends on the value of *k*. When *i, j, k* are given and satisfy the conditions (6), (6′), (6″), we can obtain all the terms by giving to *u, v, w,* in succession all integral values consistent with the mutually compatible conditions

$$\begin{aligned} &u \nless 1,\ u \nless i+\alpha_0,\ u \ngtr p,\ u \ngtr j-\beta_0-\gamma_0-\delta_0\ ; \\ &v \nless 1,\ v \nless u+\beta_0,\ v \ngtr q,\ v \ngtr j-\gamma_0-\delta_0\ ; \\ &w \nless 1,\ w \nless v+\gamma_0.\ w \ngtr r,\ w \ngtr j-\gamma_0\ ; \end{aligned} \qquad \ldots \quad (1)$$

or to *w, v, u,* in succession all integral values consistent with the mutually compalible conditions

$$\begin{aligned} &w \nless 1,\ w \nless i+\alpha_0+\beta_0+\gamma_0,\ w \ngtr r,\ w \ngtr j-\delta_0\ ; \\ &v \nless 1,\ v \nless i+\alpha_0+\beta_0, \qquad v \ngtr q,\ v \ngtr w-\gamma_0\ ; \\ &u \nless 1,\ u \nless i+\alpha_0, \qquad\quad u \ngtr p,\ u \ngtr v-\beta_0. \end{aligned} \qquad \ldots \quad (1')$$

When $\rho \ngtr m+n-1$, *i.e.* when $\xi=\rho$ and $\xi_0=n-\rho$, the elements of the first effective parametric diagonal line of X, *i.e.* the elements of X having the least possible difference-weight ξ_0 of a non-zero element, are given by the formula

$$x_{ij}=\Sigma a_{iu}\, b_{uv}\, c_{vw}\, d_{wj}, \qquad (j-i=\xi_0=n-\rho), \qquad \ldots \quad (1_0)$$

where the possible values of i and j are those consistent with the conditions

$$i \nless 1-\xi_0,\ i \ngtr m\ ; \quad j \nless 1,\ j \ngtr m+\xi_0\ ;$$

and where $\quad u=i+\alpha_0=v-\beta_0,\ v=u+\beta_0=w-\gamma_0,\ w=v+\gamma_0=j-\delta_0,$

or

$$\begin{aligned} u&=i+\alpha_0 & &=j-\beta_0-\gamma_0-\delta_0, \\ v&=i+\alpha_0+\beta_0 & &=j-\gamma_0-\delta_0, \\ w&=i+\alpha_0+\beta_0+\gamma_0 & &=j-\delta_0, \end{aligned}$$

there being only one term in the sum (*b*). In this case we have

$$\xi_0=\alpha_0+\beta_0+\gamma_0+\delta_0.$$

Theorem I shows that :

The parametric rank ξ *of* X *is that integer nearest in value to* ρ *which satisfies the necessary conditions*

$$\xi \nless 0,\ \xi \ngtr m+n-1.$$

The least difference-weight ξ_o *of a non-zero element of* X *cannot be less then the sum* $n-\rho$ *of the least difference-weights of non-zero elements of* $A, B, C, D,$ *and is that integer nearest in value to* $n-\rho$ *which satisfies the necessary conditions*

$$\xi_o \nless 1-m,\ \xi_o \ngtr n-1.$$

When A, B, C, D are particularised, their nullities remaining unaltered, these statements give respectively the greatest possible value of ξ and the least possible value of ξ_o.

6. Some auxiliary theorems.

The lemmas to be now given are used in obtaining the result of the next article. In each of them it is to be understood that $A=[a]_m^n$ is a most general direct simple matrix of any non-ruled type θ having zero horizontal and vertical nullities and given parametric rank α, so that α is a positive integer whose possible values are those consistent with the conditions:

$$\alpha \nless m,\ \alpha \nless n,\ \alpha \ngtr m+n-1.$$

The lemmas remain true when θ is any ruled type provided that we make no distinction between the two types $\pm\,\theta$, *i. e.* between two types such as $\{\pi, -\pi'\}$, $\{-\pi, \pi'\}$ each of which is derived from the other by changing the signs of both its elements. It is of course to be understood as always that $m \nless 1$ and $n \nless 1$.

Lemma A. *Let* $(A')=[a']_{m-x}^n$ *be the matrix formed by striking out the first* x *horizontal parametric rows of* A*, where*

$$x \nless 0,\ x \ngtr m-1\ ; \quad \textit{and therefore} \quad x \ngtr \alpha-1.$$

Then A' *is a most general non-zero simple matrix of type* θ *having parametric rank* $\alpha-x$*, horizontal nullity* 0*, vertical nullity* k*,*

where k *is the greater of the two integers* 0 *and* $x-(\alpha-n)$*,*

so that $k=0$ *when* $x \ngtr \alpha-n$*,* $k=x-(\alpha-n)$ *when* $x \nless \alpha-n$*.*

It is sufficient to observe that each time an initial horizontal parametric row is struck out, the parametric rank is reduced by 1 ; and that the number of initial horizontal parametric rows in A which contain no pre-parametric zero elements is $\alpha+1-n$. In the excluded case when $x=n$, the matrix A′ is non-existent.

When A is of the ordinary type $\{\pi, \pi\}$, we can represent it and A′ by hemipteric class-symbols of the first type in the forms

$$\mathrm{A}=\begin{pmatrix} n \\ m \end{pmatrix}=\begin{pmatrix} k, & n-k \\ x, & m-x \end{pmatrix}, \qquad \mathrm{A}'=\begin{pmatrix} k, & n-k \\ & m-x \end{pmatrix},$$

where the effective constituent $\mathrm{A}_1=\begin{pmatrix} n-k \\ m-x \end{pmatrix}$ of A′ is a most general simple marix having zero horizontal and vertical nullities and parametric rank $\alpha-x$, A_1 being a simple pre-major slope when $k \not< 0$, *i.e.* when $x \not< \alpha-n$. If we put

$$\mathrm{A}=\begin{bmatrix} u, & \times \\ 0, & a' \end{bmatrix}_{n,\ m-x}^{k,\ n-k}, \quad \mathrm{A}'=[0,\ a']_{m-x}^{k,\ n-k}, \quad \mathrm{A}_1=[a']_{m-x}^{n-k},$$

the first diagonal constituent U of A (when it exists, *i. e.* when $k>0$, or $x>\alpha-n$) is a general simple post-major slope of type θ, having parametric rank $x=\alpha-(n-k)$ and zero horizontal and vertical nullities. When A is a general simple pre-major or post-major or non-major slope, the same is true of A_1.

Lemma B. *Let $A''=[a'']_{m}^{n-y}$ be the matrix formed by striking out the last y vertical parametric rows of A, where*

$$y \not< 0, \ y \not> n-1 \ ; \qquad \textit{so that} \quad y \not> \alpha-1.$$

Then A'' is a most general non-zero simple matrix of type θ having parametric rank $\alpha-y$, horizontal nullity h, vertical nullity 0,

where h is the greater of the two integers 0 and $y-(\alpha-m)$,

so that $h=0$ when $y \not> \alpha-m$, $h=y-(\alpha-m)$ when $y \not< \alpha-m$.

It is sufficient to observe that each time a final vertical parametric row is struck out, the rank is reduced by 1 ; and that the number of final vertical parametric rows in A which contain no post-parametric zero elements is $\alpha+1-m$. In the excluded case when $y=n$, A″ is non-existent.

When A is of the ordinary type $\{\pi, \pi\}$, we can represent it and A$''$ by hemipteric class-symbols of the first type in the forms

$$\mathrm{A}=\begin{pmatrix} n \\ m \end{pmatrix}=\begin{pmatrix} n-y, & y \\ m-h, & h \end{pmatrix}, \qquad \mathrm{A}''=\begin{pmatrix} n-y & \\ m-h, & h \end{pmatrix},$$

where the effective constituent $\mathrm{A}_2=\begin{pmatrix} n-y \\ m-h \end{pmatrix}$ of A$''$ is a most general simple matrix having zero horizontal and vertical nullities and parametric rank $\alpha-y$, A$_2$ being a general simple post-major slope when $h \not< 0$, *i.e.* when $y \not< \alpha-m$. If we put

$$\mathrm{A}=\begin{bmatrix} a'', & \times \\ 0, & v \end{bmatrix}_{m-h,\ h}^{n-y,\ y}, \quad \mathrm{A}''=\begin{bmatrix} a'' \\ 0 \end{bmatrix}_{m-h,\ h}^{n-y}, \quad \mathrm{A}_2=[a'']_{m-h}^{n-y},$$

the last diagonal constituent V of A (when it exists, *i.e.* when $h>0$, or $y>\alpha-m$) is a general simple pre-major slope of type θ, having parametric rank $y=\alpha-(m-h)$ and zero horizontal and vertical nullities. When A is a general simple pre-major or post-major or non-major slope, the same is true of A$_2$.

Lemma C. *Let* $C=[c]_{m-y}^{n-y}$ *be the matrix formed by striking out the first* x *horizontal and last* y *vertical parametric rows of A, where* $x \not< 0$, $x \not> m-1$; $y \not< 0$, $y \not> n-1$; $x+y \not> \alpha-1$; *so that* $x+y \not> m+n-1$. *Then C is a most general non-zero simple matrix of type* θ *having*

parametric rank $\alpha-x-y$, *horizontal nullity* h, *vertical nullity* k,

where *h is the greater of the two integers 0 and* $y-(\alpha-m)$,

and *k is the greater of the two integers 0 and* $x-(\alpha-n)$.

This can be proved by successive applications of Lemmas A and B. In the excluded cases when $x=m$ or $y=n$, C is non-existent. In the excluded case when $x+y>\alpha-1$, C is a zero matrix.

When A is of the ordinary type $\{\pi, \pi\}$, we can represent A and C by hemipteric class-symbols of the first type in the forms

$$\mathrm{A}=\begin{pmatrix} n \\ m \end{pmatrix}=\begin{pmatrix} k, & n-k-y, & y \\ x, & m-h-x, & h \end{pmatrix}, \quad \mathrm{C}=\begin{pmatrix} k, & n-k-y & \\ & m-h-x, & h \end{pmatrix};$$

where the effective constituent $\mathrm{C}_0=\begin{pmatrix} n-k-y \\ m-h-a \end{pmatrix}$ of C is a most general simple matrix having zero horizontol and vertical nullities and parametric rank $\alpha-x-y$, C_0 being a general simple pre-major slope when $x \nless \alpha-n$ or $h=x-(\alpha-n)$, a general simple post-major slope when $y \nless \alpha-m$ or $h=y-(\alpha-m)$, and a general simple non-major slope when both $x \nless \alpha-n$ and $y \nless \alpha-m$. If we put

$$A=\begin{bmatrix} a, & \times, & \times \\ 0, & c, & \times \\ 0, & 0, & v \end{bmatrix}^{k,\ n-k-y,\ y}_{x,\ m-x-h,\ h}, \quad C=\begin{bmatrix} 0, & c \\ 0, & 0 \end{bmatrix}^{k,\ n-k-y}_{m-x-h,\ h},$$

$$C_o=[c]^{n-k-y}_{m-x-h},$$

the first diagonal constituent U of A (when it exists, *i.e.* when $k>0$) is a general simple post-major slope of type θ, and the last diagonal constituent V (when it exists, *i.e.* when $h>0$) is a general simple pre-major slope of type θ. When A is a general simple pre-major or post-major or non-major slepe, the same is true of C_o.

7. The product matrix in a commutantal product of most general simple matrices having given horizontal and vertical nullities and given parametric ranks.

In each of the factor matrices all three of the nullities (basical, horizontal, vertical) are given, but instead of regarding the basical nullities as given, it is more convenient to regard the parametric (or apical) ranks as given. Accordingly we will now consider the commutantal product

$$[a]^{p}_{m}\ [b]^{q}_{p}\ [c]^{r}_{q}\ [d]^{n}_{q}=[x]^{n}_{m} \quad \text{or} \quad ABCD=X \qquad \ldots \quad (C)$$

in which A, B, C, D are four most general simple matrices expressed in the standard direct forms appropriate to their types and having given parametric ranks

$$\alpha,\ \beta,\ \gamma,\ \delta, \qquad \ldots \quad (\alpha_o)$$

given horizontal nullities $\alpha_1, \beta_1, \gamma_1, \delta_1$, and given vertical nullities $\alpha_2, \beta_2, \gamma_2, \delta_2$. The post-major and pre-major parametric excesses of their effective constituents are the essentially positive integers

$$\alpha_{10}=\alpha+\alpha_1-m,\ \beta_{10}=\beta+\beta_1-p,\ \gamma_{10}=\gamma+\gamma_1-q,\ \delta_{10}=\delta+\delta_1-r \ldots \quad (1)$$

$$\alpha_{20}=\alpha+\alpha_2-p,\ \beta_{30}=\beta+\beta_2-q,\ \gamma_{20}=\gamma+\gamma_2-r,\ \delta_{20}=\delta+\delta_2-n; \ldots \quad (1')$$

their basical nullities are the essentially positive integers, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$, given by

$$\alpha+\bar{\alpha}=m+p-1,\ \beta+\bar{\beta}=p+q-1,\ \gamma+\bar{\gamma}=q+r-1,\ \delta+\bar{\delta}=r+n-1,$$

and their least effective difference-weights are the integers

$$\alpha_0=p-\alpha,\quad \beta_0=q-\beta,\quad \gamma_0=r-\gamma,\quad \delta_0=n-\delta \qquad \ldots \quad (1'')$$

each of which may be either positive or negative. The product (C) is derived from the product (A) of Art. 4 by putting all elements of A, B, C, D lying in their first $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$, parametric diagonal lines (counting from the base), their last α_1, β_1, γ_1, δ_1, horizontal parametric rows, and their first α_2, β_2, γ_2, δ_2 vertical parametric rows equal to 0, the parameters of their other elements (whose difference-weights cannot be less than α_0, β_0, γ_0, δ_0 respectively) remaining arbitrary and independent. The product matrix X is to be regarded as having the type which renders the equation (C) commutantal, and is supposed to be expressed in the direct standard form appropriate to that type. The parametric rank, horizontal nullity and vertical nullity of X (which have to be determined) will be denoted by ξ, ξ_1 and ξ_2, these being essentially positive integers. Then the post-major and pre-major parametric excesses of the least effective constituent of X will be the essentially positive integers

$$\xi_{10}=\xi+\xi_1-m,\quad \xi_{20}=\xi+\xi_2-n \qquad \ldots \quad (2)$$

and the basical nullity and least effective difference-weight of X will be the integers $\bar{\xi}$ and ξ_0 given by

$$\xi+\bar{\xi}=m+n-1,\quad \xi_0=n-\xi,$$

$\bar{\xi}$ being essentially positive, and ξ_0 being subject to the conditions

$$\xi_0 \nless 1-m,\quad \xi_0 \ngtr n.$$

If A is a zero matrix; then X is also a zero matrix, and we have

$$\alpha=0,\ \alpha_1=m,\ \alpha_2=p,\ \alpha_{10}=0,\ \alpha_{20}=0,\ \alpha_0=p,\ \alpha-\alpha_{10}-\alpha_{20}=0. \quad \ldots \quad (3)$$

If A is not a zero matrix, the possible values of α, α_1, α_2 are those consistent with the condtions

$$\alpha_1 \nless 0,\ \alpha_2 \nless 0,\ \alpha+\alpha_1 \nless m,\ \alpha+\alpha_2 \nless p,\ \alpha+\alpha_1+\alpha_2 \ngtr m+p-1; \quad \ldots \quad (4)$$

which include $\alpha \nless 1$, $\alpha \ngtr m+p-1$; $\alpha_1 \nless 0$, $\alpha_2 \ngtr m-1$, $\alpha_2 \nless 0$, $\alpha_2 \ngtr p-1$; the possible values of α, α_{10}, α_{20} are those consistent with the conditions

$$\alpha_{10} \nless 0,\ \alpha_{10} \nless \alpha-m,\ \alpha_{20} \nless 0,\ \alpha_{20} \nless \alpha-p,\ \alpha_{10}+\alpha_{20} \ngtr \alpha-1; \ldots \quad (4')$$

which include

$$\alpha \nless 1,\ \alpha \ngtr m+p-1;\ \alpha_{10} \nless 0,\ \alpha_{10} \ngtr p-1,\ \alpha_{20} \nless 0,\ \alpha_{20} \ngtr m-1;$$

and it is to be observed in particular that we have

$$\alpha \nless 1,\ \alpha-\alpha_{10}+\alpha_{20} \ngtr 1. \qquad \dots \quad (4'')$$

We shall usually regard α_1, α_2, $\overline{\alpha}$, α_0 as being expressed in terms of α, α_{10} α_{20}.

We have corresponding results for B, C, D, which show that two corresponding integers in the set (a_0) and the set

$$\alpha-\alpha_{20}-\alpha_{10},\ \beta-\beta_{20}-\beta_{10},\ \gamma-\gamma_{20}-\gamma_{10},\ \delta-\delta_{20}-\delta_{10} \qquad \dots \quad (a_1)$$

are either both equal to 0 or both greater than 0 ; and that when A, B, C, D are all non-zero matrices, all the integers (a_0) and (a_1) are greater than 0. The product matrix only needs evaluation when A, B, C, D are non-zero matrices, so that the conditions (4), (4′) (4″) and the corresponding conditions for B, C, D are all satisfied ; and this is the case when and only when all the integers (a_0) and (a_1) are greater than 0.

If for the sake of brevity we put

$$\rho_{\alpha\beta}=\alpha+\beta-p,\ \rho_{\beta\gamma}=\beta+\gamma-q,\ \rho_{\gamma\delta}=\gamma+\delta-r,$$

$$\rho_{\alpha\beta\gamma}=\alpha+\beta+\gamma-q,\ \rho_{\beta\gamma\delta}=\beta+\gamma+\delta-r,$$

$$\rho=\rho_{\alpha\beta\gamma\delta}=\alpha+\beta+\gamma+\delta-p-q-r, \qquad \dots \quad (5)$$

it will be found that the general character of X depends on the values of the integers

$$\rho_{\alpha\beta}-\alpha_{20}-\beta_{10},\ \rho_{\beta\gamma}-\beta_{20}-\gamma_{10},\ \rho_{\gamma\delta}-\gamma_{20}-\delta_{10}, \qquad \dots \quad (a_2)$$

$$\rho_{\alpha\beta\gamma}-\alpha_{20}-\gamma_{10},\ \rho_{\beta\gamma\delta}-\beta_{20}-\delta_{10}, \qquad \dots \quad (a_3)$$

$$\rho_{\alpha\beta\gamma\delta}-\alpha_{20}-\delta_{10}=\rho-\alpha_{20}-\delta_{10}, \qquad \dots \quad (a_4)$$

which will be called the *discriminators* of the partial products AB, BC, CD, ABC, BCD, ABCD formed with consecutive factors of the product (C), and the values of the integers

$$\alpha-\alpha_{10},\ \rho_{\alpha\beta}-\beta_{10},\ \rho_{\alpha\beta\gamma}-\gamma_{10},\ \rho_{\alpha\beta\gamma\delta}-\delta_{10}, \qquad \dots \quad (b_1)$$

$$\delta-\delta_{20},\ \rho_{\gamma\delta}-\delta_{20};\ \rho_{\beta\gamma\delta}-\beta_{20},\ \rho_{\alpha\beta\gamma\delta}-\alpha_{20}, \qquad \dots \quad (b_2)$$

which, if $\qquad I=[1]_m^m$ and $J=[1]_n^n$,

so that $$X=IABCDJ,$$
are the discriminators of the partial products

IA, IAB, IABC, IABCD, and DJ, CDJ, BCDJ, ABCDJ.

The integer ρ is the discriminator of the product IABCDJ.

In fact if, when A, B, C, D, are all non-zero matrices, we first:

cancel in succession the (passive) initial vertical zero major-parametric rows of A, B, C with the corresponding (passive) initial horizontal major-parametric rows of B, C, D, applying Lemma A of Art. **6** to the effective constituents of B, C, D when they are not thereby annihilated, and stopping as soon as the annihilation of one of them shows X to be a zero matrix;

and, when X has not been thus shown to be a zero matrix, next:

cancel in succession the (passive) final horizontal zero major-parametric rows of D, C, B with the corresponding (passive) final vertical major-parametric rows of C, B, A, applying Lemma B to the effective constituents of C, B, A, which will never be annihilated;

we shall obtain the following theorem :

Theorem I. *Let θ be the symbolic commutant which is the product of the symbolic commutants representing the types of the successive factor matrices A, B, C, D in the commutantal product* (C). *Then in evaluating the product matrix X we can distinguish between two cases :*

CASE I. *If any one of the discriminators (a_2), (a_3), (a_4), or if we please any one of the discriminators (a_1), (a_2), (a_3), (a_4), is not greater than 0, X must be a zero matrix.*

CASE II. *If every one of the discriminators (a_2), (a_3), (a_4), or if we please every one of the discriminators (a_1), (a_2), (a_3), (a_4), is greater than 0, then X is non-zero simple matrix of the type θ having horizontal nullity ξ_1, vertical nullity ξ_2, and parametric rank ξ, where :*

(1) *$m-\xi_1$ is the smallest of the non-zero poitive integers (b_1) (the condition $m-\xi_1 \not> a-a_{10} \not> m-a_1$ showing that ξ_1 is positive) ;*

(2) *$n-\xi_2$ is the smallest of the non-zero positive integers (b_2), (the condition $n-\xi_2 \not> \delta-\delta_{10} \not> n-\delta_2$ showing that ξ_2 is positive) ;*

(3) *ξ is the smaller of the two non-zero positive integers*
$$\rho=a+\beta+\gamma+\delta-v-q-r,\quad \sigma=(m-\xi_1)+(n-\xi_2)-1\,;$$
and none of the elements of the ξ effective parametric diagonal lines of

the least effective constituent X_0 of X vanish identically, i.e. X has the form of a most general simple matrix having horizontal nullity ξ_1, vertical nullity ξ_2, and parametric rank ξ.

NOTE—*Alternative form of Theorem I.*

Let ω be the smallest of the discriminators (a_1), (a_2), (a_3), (a_4) ; let ω_1 be the smallest of the discriminators (b_1) ; let ω_2 be the smallest of the discriminators (b_2) ; and let ρ be the discriminator of the product IABCDJ ; so that

$$\rho \nless \omega_1 \nless \omega, \quad \rho \nless \omega_2 \nless \omega.$$

Then if $\omega \ngtr 0$, X is zero matrix ; and if $\omega > 0$, X is a non-zero simple matrix of type θ having horizontal nullity $\xi_1 = m - \omega_1$, vertical nullity $\xi_2 = n - \omega_2$, and parametric rank ξ, where ξ is that integer nearest in value to ρ which (see Ex. iii. of Art. **3**) satisfies the necessary condition

$$\xi \ngtr \omega_1 + \omega_2 - 1.$$

We will consider first the trivial case in which one of the factor matrices A, B, C, D is a zero matrix, *i.e.* one of the discriminators $(a\ )$ is not greater than 0, so that X is necessarily a zero matrix. Let P and Q be any two consecutive factor matrices. Then if either P or Q is a zero matrix, *i.e.* if the discriminator of either P or Q is not greater than 0, the discriminator of the product PQ cannot be greater than 0. For example if B is a zero matrix, we have $\beta = 0$, $\beta_{10} = 0$, $\beta_{20} = 0$, and therefore

$$\rho_{\alpha\beta} - \alpha_{20} - \beta_{10} = (\alpha + \beta - p) - \alpha_{20} - \beta_{10} = \alpha - \alpha_{20} - p = -\alpha_2 \ngtr 0,$$

$$\rho_{\beta\gamma} - \beta_{20} - \gamma_{10} = (\beta + \gamma - q) - \beta_{20} - \gamma_{10} = \gamma - \gamma_{10} - q = -\gamma_1 \ngtr 0,$$

i.e. the discriminators of the products AB, BC, cannot be greater than 0. These considerations show that Theorem I is true when A, B, C, D are not all zero matrices, Case I only being possible. They show further that every one of the discriminators (a_2), (a_3), (a_4) is greater than 0 when and only when every one of the discriminators (a_1), (a_2), (a_3), (a_4) is greater than 0, and also when and only when every one of the integers (a_0), (a_1), (a_2), (a_3), (a_4) is greater than 0. Hence in proving the theorem we may suppose the discriminators mentioned in it to be (a_2), (a_3), (a_4) ; but it makes no difference if we add to them the discriminators (a_1) or the discriminators (a_1) and the integers (a_0).

Again if the theorem is true for the correlated commutantal product A′B′C′D′=X′ in which A′, B′, C′, D′ are all of the ordinary type $\{\pi, \pi\}$, it will be true for the given commutantal product (C). For then X in the product (C) will be the simple matrix of type θ correlated with the simple matrix X′ of type $\{\pi, \pi\}$, and X will have the same orders, the same rank, same horizontal and vertical nullities, and the same parametric rank as X′. Moreover the expression for any parameter $x_{i,j}$ of X as a rational integral function of the parameters of A, B, C, D will be identical with the expression for the parameter $x_{i,j}$ of X′ as a rational integral function of the parameters of A′, B′, C′, D′.

Thus in proving Theorem. I and the following theorems of this article, we may suppose that A, B, C, D are all non-zero matrices, so that all the conditions such as (4), (4′), (4″), are satisfied ; and we may further suppose that A, B, C, D are all of the ordinary type $\{\pi, \pi\}$.

When those suppositions are made, we have to determine X from an equation of the form

$$X = ABCD = \begin{bmatrix} 0, & a' \\ 0, & 0 \end{bmatrix}^{\alpha_2,\, p-\alpha_2}_{m-\alpha_1,\, \alpha_1} \begin{bmatrix} 0, & b' \\ 0, & 0 \end{bmatrix}^{\beta_2,\, q-\beta_2}_{p-\beta_1,\, \beta_1}$$

$$\begin{bmatrix} 0, & c' \\ 0, & 0 \end{bmatrix}^{\gamma_2,\, r-\gamma_2}_{q-\gamma_1,\, \gamma_1} \begin{bmatrix} 0, & d' \\ 0, & 0 \end{bmatrix}^{\delta_2,\, n-\delta_2}_{r-\delta_1,\, \delta_1},$$

or, when we represent A, B, C, D by hemipteric class-symbols of the first type, from an equation of the form

$$X = ABCD = \begin{pmatrix} \alpha_2, \; p-\alpha_2 & \\ & m-\alpha_1, \; \alpha_1 \end{pmatrix} \begin{pmatrix} \beta_2, \; q-\beta_2 & \\ & p-\beta_1, \; \beta_1 \end{pmatrix} \begin{pmatrix} \gamma_2, \; r-\gamma_2 & \\ & q-\gamma_1, \; \gamma_1 \end{pmatrix}$$

$$\begin{pmatrix} \delta_2, \; n-\delta_2 & \\ & r-\delta_1, \; \delta_1 \end{pmatrix},$$

in which the effective constituents of A, B, C, D are most general non-zero simple matrices of the type $\{\pi, \pi\}$ having zero horizontal and vertical nullities and the given parametric ranks

$$\alpha, \quad \beta, \quad \gamma, \quad \delta.$$

The successive cancellations mentioned above can now be easily carried out by utilising the properties of hemipteric class-symbols of the first type, the cancellations of corresponding passive rows being represented by cancellations of the corresponding index numbers.

When it is necessary to carry through all the cancellations of the first set, we cancel in succession the α_1, P_2, Q_2 initial vertical rows of 0's in A, B, C with the corresponding initial horizontal rows of

B, C, D, thus reducing the equation determining X to the form

$$X=A'B''C''D'=\begin{pmatrix} p-\alpha_2 & \\ m-\alpha_1, & \alpha_1 \end{pmatrix} \begin{pmatrix} q-P_2 & \\ p-\alpha_2-\beta_1, & \beta_1 \end{pmatrix} \begin{pmatrix} r-Q_2 & \\ q-P_2-\gamma_1, & \gamma_1 \end{pmatrix}$$

$$\begin{pmatrix} R_2, & n-h \\ & r-Q_2-\delta_1, \ \delta_1 \end{pmatrix},$$

where $P_2=p_2+\beta_2$ is the greater of the two integers $\beta_2, \alpha_2+(q-\beta)$,

$Q_2=q_2+\gamma_2$ is the greater of the two integers γ_2, $P_2+(r-\gamma)$,

$R_2=r_2+\delta_2$ is the greater of the two integers δ_2, $Q_2+(n-\delta)$,

and where the integers p_2, q_2, r_2 thus defined are the numbers of the additional initial vertical rows of 0's which the successive cancellations introduce into B, C, D. The successive cancellations show that X is necessarily a zero matrix unless the successive conditions

$$p-\alpha_2-\beta_1>0,\ q-P_2-\gamma_1>0,\ r-Q_2-\delta_1>0$$

are all satisfied ; and since (using the definitions of P_2 and Q_2)

$p-\alpha_2-\beta_1$ is the discriminator of AB

$q-P_2-\gamma_1$ is the smaller of the discriminators of BC, ABC,

$r-Q_2-\delta_1$ is the smallest of the discriminators of CD, BCD, ABCD,

it follows that Theorem I is true as regards Case I. When all these conditions are satisfied, the effective constituents of A′, B″, C″, D′ are most general non-zero simple matrices of the type $\{\pi, \pi\}$ having zero horizontal and vertical nullities and parametric ranks

$$\alpha,\quad \beta-\alpha_2,\quad \gamma-P_2,\quad \delta-Q_2.$$

When the discriminators (α_2), (α_3), (α_4) are all greater than 0, and all the preceding cancellations have been carried through, we proceed to the second set of cancellations. We cancel in succession the δ_1, R_1, Q_1 final horizontal rows of 0's in D′, C″, B″ with the corresponding final vertical rows of C″, B″, A′, thus reducing the equation determining X to the form

$$X=A_1\,B_0\,C_0\,D_0$$

$$=\begin{pmatrix} p-\alpha_2-Q_1 & \\ m-P_1, & P_1 \end{pmatrix} \begin{pmatrix} q-P_2-R_1 & \\ p-\alpha_1-Q_1 & \end{pmatrix} \begin{pmatrix} r-Q_2-\delta_1 & \\ q-P_2-R_1 & \end{pmatrix} \begin{pmatrix} R_2, & n-R_2 \\ & r-Q_2-\delta_1 \end{pmatrix}$$

where $R_1 = r_1 - \gamma_1$ is the greater of the two integers

$$\gamma_1,\ (q-\gamma)+\delta_1,$$

$Q_1 = q_1 + \beta_1$ is the greater of the two integers

$$\beta_1,\ (p-\beta)+R_1,$$

$P_1 = p_1 + \alpha_1$ is the greater of the two integers

$$\alpha_1,\ (m-\alpha)+Q_1,$$

and where the integers r_1, q_1, p_1 thus defined are the number of the additional final horizontal rows of 0's which the successive cancellations introduce into C″, B″, A′. The successive cancellations show that X is necessarily a zero matrix unless the successive conditions

$$r-Q_2-\delta_1>0,\quad q-P_2-R>0,\quad p-\alpha_2-Q_1>0$$

are all satisfied ; but since

$r-Q_2-\delta_1$ is the smallest of the discriminators of

CD, BCD, ABCD,

$q-P_2-R_1$ is the smallest of the discriminators of

BC, ABC, BCD, ABCD,

$p-\alpha_2-Q_1$ is the smallest of the discriminators of

AB, ABC, ABCD,

these conditions are all satisfied by supposition. After these cancellation have been completed, the effective constituents A_0, B_0, C_0, D_0 of A_1, B_0, C_0, D_2 are most general non-zero simple matrices of the type $\{\pi, \pi\}$ having zero horizontal and vertical nullities and parametric ranks

$$\alpha-Q_1,\quad \beta-\alpha_2-R_1,\quad \gamma-P_2-\delta_1,\quad \delta-Q_2.$$

Thus if we put $\xi_1 = P_1$, $\xi_2 = R_2$,

and $X = ABCD = IABCDJ$, where $I = [1]_m^m$ and $J = [1]_n^n$,

we see that in Case II of Theorem I the product matrix X is determined by the equation

$$X = A_1 B_0 C_0 D_2 = \begin{pmatrix} p_0 & \\ m-\xi_1, & \xi_1 \end{pmatrix} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \begin{pmatrix} r_0 \\ q_0 \end{pmatrix} \begin{pmatrix} \xi_2, & n-\xi_2 \\ & r_0 \end{pmatrix}$$

$$= \begin{pmatrix} p_0 & \\ m_0, & P_1 \end{pmatrix} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \begin{pmatrix} r_0 \\ q_0 \end{pmatrix} \begin{pmatrix} R_2, & n_0 \\ & r_0 \end{pmatrix}, \quad \dots \quad \text{(D)}$$

where

$m_0=m-\xi_1=m-P_1$ is the smallest of the integers (b_1) which are the discriminators of the partial products

IA, IAB, IABC, IABCD containing IA,

$p_0=p-\alpha_2-Q_1$ is the smallest of the discriminators of the partial products

AB, ABC, ABCD containing AB,

$q_0=q-P_2-R_1$ is the smallest of the discriminators of the partial products

BC, ABC, BCD, ABCD containing BC,

$r_0=r-Q_2-\delta_1$ is the smallest of the discriminators of the partial products

CD, BCD, ABCD, containing CD,

$n_0=n-\xi_2=n-R_2$ is the smallest of the integers (b_1) which are the discriminators of the partial products

DJ, CDJ, BCDJ, ABCDJ containing DJ ;

all these discriminators being non-zero positive integers ; and where

$$A_0=\begin{pmatrix}p_0\\ m-\xi_1\end{pmatrix}=\begin{pmatrix}p_0\\ m_0\end{pmatrix},\quad B_0=\begin{pmatrix}q_0\\ p_0\end{pmatrix},$$

$$C_0=\begin{pmatrix}r_0\\ q_0\end{pmatrix},\quad D_0=\begin{pmatrix}n_0\\ r_0\end{pmatrix}=\begin{pmatrix}n-\xi_2\\ r_0\end{pmatrix}$$

are most general non-zero simple matrices of the type $\{\pi, \pi\}$ having zero horizontal and vertical nullities and parametric ranks

$$\alpha-Q_1,\quad \beta-\alpha_2-R_1,\quad \gamma-P_2-\delta_1,\quad \delta-Q_2.$$

The integers ξ_1 and ξ_2 are those defined in Theorem I.

Since X as given by the formula (D) has ξ_1 final horizontal and ξ_2 initial vertical rows of 0's we can write

$$X=\begin{bmatrix}0, & x''\\ 0, & 0\end{bmatrix}^{\xi_2,\ n-\xi_2}_{m-\xi_1,\ \xi_1}=\begin{pmatrix}\xi_2,\ n-\xi_2\\ m-\xi_1,\ \xi_1\end{pmatrix},$$

$$X_0=[x'']^{n-\xi_2}_{m-\xi_1}=\begin{pmatrix}n-\xi_2\\ m-\xi_1\end{pmatrix},$$

and replace (D_1) by the equation

$$\begin{pmatrix} \xi_2, & n-\xi_2 \\ & m-\xi_1, \xi_1 \end{pmatrix} = \begin{pmatrix} p_0 & \\ m-\xi_1, & \xi_1 \end{pmatrix} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \begin{pmatrix} r_0 \\ q_0 \end{pmatrix} \begin{pmatrix} \xi_2, & n-\xi_2 \\ & r_0 \end{pmatrix}$$

or

$$\begin{bmatrix} 0, & x'' \\ 0, & 0 \end{bmatrix}^{\xi_2,\ n-\xi_2}_{m-\xi_1,\ \xi_1} = \begin{bmatrix} a'' \\ 0 \end{bmatrix}^{p_0}_{m-\xi_1,\ \xi_1} [b'']^{q_0}_{p_0} [c'']^{r_0}_{q_0} [0, d'']^{\xi_2,\ n-\xi_2}_{r_0} \qquad \ldots \quad (D')$$

and when we cancel corresponding active rows of 0's on both sides, we see that X_0 is the product matrix in the commutantal product

$$[a'']^{p_0}_{m-\xi_1} [b'']^{q_0}_{p_0} [c'']^{r_0}_{q_0} [d'']^{n-\xi_2}_{r_0} = [x'']^{n-\xi_2}_{m-\xi_1},$$

$$\text{or} \qquad A_0B_0C_0D_0 \qquad = X, \qquad \ldots \quad (C_0)$$

which is one of the kind considered in Art. **5**. Observing that

$$\{(\alpha-Q_1)+(\beta-\alpha_3-R_1)+(\gamma-P_2-\delta_1)+\delta-Q_2)\}-(p_0+q_0+r_0)=\rho,$$

this being a necessary consequence of the characters of the passive cancellations, it follows from Art. **5** that X_0 is a non-zero simple matrix of the type $\{\pi, \pi\}$ having zero horizontal and vertical nullities, and parametric rank ξ, where ξ is the smaller of the two integers

$$\rho, \ \sigma=(m-\xi_1)+(n-\xi_2)-1,$$

and that none of the elements forming the ξ effective parametric diagonal lines of X_0 are equal to 0. Thus Theorem I is also true as regards Case II.

In proving Theorem I we have also proved the following second theorem :

Theorem II. *In Case II of Theorem I, i.e. when the discriminators of all the products AB, BC, CD, ABC. BCD, ABCD are greater than 0, the effective constituent X_0 of the product matrix X in the commutantal product (C) is the product matrix in the commutantal product*

$$[a'']^{p_0}_{m_0} [b'']^{q_0}_{p_0} [c'']^{r_0}_{q_0} [d'']^{n_0}_{r_0} = [x'']^{n_0}_{m_0} \qquad \text{or} \qquad A_0B_0C_0D_0=X_0,$$

where A_0, B_0, C_0, D_0 *are the general non-zero simple matrices with zero horizontal and vertical nullities and with parametric ranks*

$$\alpha-Q_1,\quad \beta-\alpha_2-R_1,\quad \gamma-P_2-\delta_1,\quad \delta-Q_2$$

which are formed from A, B, C, D *by striking out from each of them as many initial vertical, initial horizontal, final vertical and final horizontal major-parametric rows as are indicated in the following table where the last column shows the number of rows of each kind which must be struck out from* X *to form* X_0 :

		A	B	C	D	X
Initial rows	*vertical*	$M_2=\alpha_2$	P_2	Q_2	$R_2=\xi_2$	ξ_2
	horizontal	——	$M_2=\alpha_2$	P_2	Q_2	—
Final rows	*vertical*	Q_1	R_1	$N_1=\delta_1$	——	—
	horizontal	$P_1=\xi_1$	Q_1	R_1	$N_1=\delta_1$	ξ_1

If we suppose A_0, B_0, C_0, D_0, X_0 to be expressed in the standard direct forms appropriate to their types, and denote their least effective difference-weights under these circumstance by α_0'', β_0'', γ_0'', δ_0'', ξ_0'' so that

$$\xi_0'' \text{ is the greater of the two integers } \quad n_0-\rho,\quad 1-m_0,$$

$$\alpha_0''=p_0-\alpha-Q_1=\alpha_0-\alpha_2,\quad \beta_0''=q_0-\beta+\alpha_2+R_1=\beta_0+\alpha_2-P_2,$$

$$\gamma_0''=r_0-\gamma+P_2+\delta_1=\gamma_0+P_2-Q_2,\quad \delta_0''=n_0-\delta+Q_2=\delta_0+Q_2-R_2,$$

$$\xi_0''=n_0-\xi=\xi_0-\xi_2,$$

we know from Art. 5 that if a''_{IJ} is an element of one of the ξ effective parametric diagonal lines of X_0, *i.e.* a non-zero element of X_0, the possible values of I and J, the possible values of the difference-weight $k''=J-I$, and the possible values of I and J when k'' is given are those consistent with the respective sets of conditions

$$I \nless 1,\ I \ngtr m_0,\ J \nless 1,\ J \ngtr n_0,\ J-I \nless n_0-\rho\ ;$$

$$k'' \nless \xi''_0,\ k'' \ngtr n_0-1\ ;$$

$$I \nless 1,\ I \nless 1-k'',\ I \ngtr n_0-k'',\ I \ngtr m_0\ ;\ J=I+k'',$$

and we know further that when I, J, k'' have any given values consistent with these conditions, the expression for the parameter x''_{IJ} is

$$x''_{\mathrm{IJ}} = \Sigma a''_{\mathrm{IU}} \, b''_{\mathrm{VU}} \, c''_{\mathrm{VW}} \, d''_{\mathrm{WJ}}, \qquad (\mathrm{J}-\mathrm{I}=k''),$$

where the summation extends over all distinct sets of values of U, V W which can be formed which U, V, W are elements of the respective sequences

$$[1, 2, \ldots p_0], \quad [1, 2, \ldots q_0], \quad [1, 2, \ldots r_0]$$

which make every term on the left positive in the equation

$$\{(\mathrm{U}-\mathrm{I})-\alpha_0''\}+\{(\mathrm{V}-\mathrm{U})-\beta_0''\}+\{(\mathrm{W}-\mathrm{V})-\gamma_0''\}+\{(\mathrm{J}-\mathrm{W})-\delta_0''\}$$
$$=k''-(n_0-\rho$$

Now let A, B, C, D, X be expressed in the standard direct forms appropriate to their types as in the previous theorems. Then

$$a''_{\mathrm{IU}} = a_{iu}, \; b''_{\mathrm{UV}} = b_{uv}, \; c''_{\mathrm{VW}} = c_{vw}, \; d''_{\mathrm{WJ}} = d_{wj},$$
$$x''_{\mathrm{IJ}} = x_{ij},$$

where

$$i=\mathrm{I}, \; u=\alpha_0+\mathrm{U}, \; v=\mathrm{P}_2+\mathrm{V}, \; w=\mathrm{Q}_2+\mathrm{W}, \; j=\mathrm{R}_2+\mathrm{J}=\xi_2+\mathrm{J} \; ;$$

and if $$k=j-i, \quad k''=\mathrm{J}-\mathrm{I}$$

are the difference-weights of the same element of X denoted by $\pm x''_{\mathrm{IJ}}$ in the former and by $\pm x_{ij}$ in the present case, we have

$$k=k''+\xi_2, \qquad \text{and in particular} \qquad \xi_0=\xi_0''+\xi_2.$$

Substituting for I, U, V, W, J, k'', ξ_0'' the values given by these equations we see that if $\pm x_{ij}$ is an element of X lying in one of the ξ effective parametric diagonal lines of X_0, *i.e.* a non-zero element of X, the possible values of i and j, the possible values of the difference-weight $k=j-i$, and the possible values of i and j when k is given are those consistent with the respective sets of conditions

$$i \nless 1, \; i \ngtr m-\xi_1, \; j \nless 1+\xi_2, \; j \ngtr n, \; j-i \nless \nu-\rho \; ; \qquad \ldots \quad (6)$$

$$k \nless \xi_0, \; k \ngtr n-1 \; ; \qquad \ldots \quad (6')$$

$$i \nless 1, \; i \nless 1+\xi_2-k, \; i \ngtr n-k, \; i \ngtr m-\xi_1 \; ; \; j=i+k \; ; \qquad \ldots \quad (6'')$$

and we have further the following theorem :

Theorem III. *The non-zero elements of X (those forming the ξ effective parametric diagonal lines of X_0) are given by the general formula*

$$x_{i,j}=\sum a_{iu}\, b_{uv}\, c_{vw}\, d_{wj}, \qquad (j-i=k), \qquad \ldots \quad (c)$$

where the possible values of i, j and k are given by (6), (6′), (6″) and where for a given pair of values of i and j the summation extends over all distinct sets of values of u, v, w, which satisfy the conditions

$$u-\alpha_2 \nless 1,\ u-\alpha_2 \ngtr p_0\ ;\ v-P_2 \nless 1,\ v-P_2 \ngtr q_0\ ;\ w-Q_2 \nless 1,\ w-Q_2 \ngtr r_0,$$

or

$$u \nless 1+\alpha_2,\ u \ngtr p-Q_1\ ;\ v \nless 1+P_2,\ v \ngtr q-R_1\ ;\ w \nless 1+Q_2,\ w \ngtr r-\delta_1,$$

and also make every term on the left positive in the equation

$$\{(u-i)-\alpha_0\}+\{(v-u)-\beta_0\}+\{(w-v)-\gamma_0\}+\{(j-w)-\delta_0\} = k-(n-\rho).$$

Thus every element of the product matrix X is known.

8. Alternative method of evaluating the product matrix X of Art. 7 when it is not a zero matrix.

We will again suppose that A, B, C, D are all of the natural type $\{\pi, \pi\}$ and are expressed in their direct standard forms. The letters which have already been introduced will have the same meanings as before, and to show more clearly the general laws, we will introduce the new letters m_2, M_2, n_2, N_2, m_1, M_1, n_1, N_1 which are defined below.

If we pay regard to the rows which were cancelled in Art. 7, but do not perform any of the cancellations, we see from Lemmas A and B of Art. 6 that when the discriminators of the products AB, BC, CD, ABC,. BCD, ABCD are all greater than 0, we can represent X as a product of hemipteric matrices of the first type in the form

$$X=ABCD=\begin{pmatrix} \alpha_2, p_0, Q_1 & \\ & m_0, p_1, \alpha_1 \end{pmatrix} \begin{pmatrix} \beta_2, p_2, q_0, R_1 & \\ & \alpha_2, p_0, q_1, \beta_1 \end{pmatrix}$$

$$\begin{pmatrix} \gamma_2, q_2, r_0, \delta_1 & \\ & P_2, q_0, r_1, \gamma_1 \end{pmatrix} \begin{pmatrix} \delta_2, r_2, n_0 & \\ & Q_2, r_0, \delta_1 \end{pmatrix}, \qquad \ldots \quad (E)$$

where m_2, M_2, p_2, P_2, q_2, Q_2, r_2, R_2, n_2, N_2 are the positive integers defined by the statements that :

$m_2=0$;

$M_2=m_2+\alpha_2=\alpha_2$;

$P_2=p_2+\beta_2$ is the greater of M_2, $M_2+(q-\beta)$, *i.e.* the greater of

$$\beta_2,\ \alpha_2+\beta_2-\beta_{20},$$

or
$$\rho_{\gamma\delta}-\{\rho_{\beta\gamma\delta}-\beta_{20}\},\ \rho_{\gamma\delta}-\{\rho_{\alpha\beta\gamma\delta}-\alpha_{20}\};$$

$Q_2=q_2+\gamma_2$ is the greater of γ_2, $P_2+(r-\gamma)$, *i.e.* the greatest of

$$\gamma_2,\ \beta_2+\gamma_2-\gamma_{20},\ \alpha_2+\beta_2+\gamma_2-\beta_{20}-\gamma_{20},$$

or
$$\delta-\{\rho_{\gamma\delta}-\gamma_{20}\},\ \ \delta-\{\rho_{\beta\gamma\delta}-\beta_{20}\};$$

$R_2=r_2+\delta_2=\xi_2$ is the greater of δ_2, $Q_2+n-\delta$, *i.e.* the greatest of

$$\delta_2,\ \gamma_2+\delta_2-\delta_{20},\ \beta_2+\gamma_2+\delta_2-\gamma_{20}-\delta_{20},\ \alpha_2+\beta_2+\gamma_2+\delta_2 \\ -\beta_{20}-\gamma_{20}-\delta_{20},$$

or
$$n-\{\delta-\delta_{20}\},\ n-\{\rho_{\gamma\delta}-\gamma_{20}\},\ n-\{\rho_{\beta\gamma\delta}-\beta_{20}\}, \\ n-\{\rho_{\alpha\beta\gamma\delta}-\alpha_{20}\};$$

$N_2=n_2=R_2=\xi$;

where n_1, N_1, r_1, R_1, q_1, Q_1, p_1, P_1, m_1, M_1, are the positive integers defined by the statements that :

$n_1=0$;

$N_1=n_1+\delta_1=\delta_1$;

$R_1=r_1+\gamma_1$ is the greater of γ_1, $(q+\gamma)+N_1$, *i.e.* the greater of

$$\gamma_1,\ \ \gamma_1+\delta_1-\gamma_{10},$$

or
$$\rho_{\alpha\beta}-\{\rho_{\alpha\beta\gamma}-\gamma_{10}\},\ \rho_{\alpha\beta}-\{\rho_{\alpha\beta\gamma\delta}-\delta_{10}\};$$

$Q_1=q_1+\beta_1$ is the greater of β_1, $(p-\beta)+R_1$, *i.e.* the greatest of

$$\beta_1,\ \beta_1+\gamma_1-\beta_{10},\ \beta_1+\gamma_1+\delta_1-\beta_{10}-\gamma_{10},$$

or
$$\alpha-\{\rho_{\alpha\beta}-\beta_{10}\},\ \ \alpha-\{\rho_{\alpha\beta\gamma}-\gamma_{10}\},\ \ \alpha-\{\rho_{\alpha\beta\gamma\delta}-\delta_{10}\};$$

$P_1=p_1+\alpha_1=\xi_1$ is the greater of α_1, $(m-\alpha)+Q_1$, *i.e.* the greatest of

$$\alpha_1,\ \alpha_1+\beta_1-\alpha_{10},\ \alpha_1+\beta_1+\gamma_1-\alpha_{10}-\beta_{10},\ \alpha_1+\beta_1+\gamma_1+\delta_1 \\ -\alpha_{10}-\beta_{10}-\gamma_{10},$$

or
$$m-\{\alpha-\alpha_{10}\},\ m-\{\rho_{\alpha\beta}-\beta_{10}\},\ m-\{\rho_{\alpha\beta\gamma}-\gamma_{10}\}, \\ m-\{\rho_{\alpha\beta\gamma\delta}-\delta_{10}\};$$

$M_1=m_1=P_1=\xi_1$;

and where $m_0=m$, p , q_0, r_0, $n=n-\xi_2$ have the values given in Art. 7.

From Lemma A of Art. **6** it will be seen that the constituents

$$\binom{p_2}{\alpha_2},\ \binom{q_2}{P_2},\ \binom{r_2}{Q_2}$$

exist respectively when and only when $p_2>0, q_2>0, r_2>0$, and that they are then general non-zero simple post-major slopes with zero horizontal and vertical nullities having the non-zero parametric ranks α_2 P_2, Q_2. When $\alpha_2=0$, then also $p_2=0$; when $P_2=0$, then also $q_2=0$; when $Q_2=0$, then also $r_2=0$. In fact

$$p_2,\quad q_2,\quad r_2$$

are respectively the greater of

$$0,\ \alpha_2-\beta_{20};\quad 0,\ P_2-\gamma_{20};\quad 0,\ Q_2-\delta_{20}.$$

From Lemma B of Art. **6** it will be seen that the constituents

$$\binom{\delta_1}{r_1},\ \binom{R_1}{q_1},\ \binom{Q_1}{p_1}$$

exist respectively when and only when $r_1>0$, $q_1>0$, $p_1>0$, and that they are then general non-zero simple pre-major slopes with zero horizontal and vertical nullities having the non-zero parametric ranks δ_1, R_1, Q_1. When $\delta_1=0$, then also $r_1=0$; when $R_1=0$, then also $q_1=0$; when $Q_1=0$, then also $p_1=0$. In fact

$$r_1,\quad q_1,\quad p_1$$

are respectively the greater of

$$0,\ \delta_1-\gamma_{10};\quad 0,\ R_1-\beta_{10};\quad 0,\ Q_1-\alpha_{10}.$$

In the formula (E) we can strike out in succession

α_2 in A, α_2 in B ; β_2 and p_2 in B, P_2 in C ;
γ_2 and q_2 in C, Q_2 in D ;
δ_1 in D, δ_1 in C ; γ_1 and r_1 in C, R_1 in B ;
β_1 and q_1 in B, Q_1 in A ;

to represent in each case cancellations of corresponding passive rows. Then X is reduced to the form (D) of Art. **7**, and the evaluation can be completed as before.

9. Generalisations and applications.

It will be immediately obvious what the general theorems for commutantal products of any number of factors are which correspond to the theorems given in Arts. **4**, **5** and **7**, and that those general theorems can be either proved directly, or proved by induction when they have been proved for products of two factors. The results obtained are of very great generality, and are applicable, with the appropriate simplifications, to many special kinds of simple matrices, both non-ruled and ruled, the most obvious applications being to simple slopes.

The direct standard forms have been used throughout, but all results obtained for the commutantal product

$$[a]^{p}_{m}\ [b]^{q}_{p}\ [c]^{r}_{q}\ [d]^{n}_{r} = [x]^{n}_{m} \quad \text{or} \quad \mathrm{ABCD} = \mathrm{X} \quad .. \quad (1)$$

remain valid for the commutantal product

$$\boxed{d}^{\,r}_{\,n}\ \boxed{c}^{\,q}_{\,r}\ \boxed{b}^{\,p}_{\,q}\ \boxed{a}^{\,m}_{\,p} = \boxed{x}^{\,m}_{\,n} \quad \text{or} \quad \mathrm{DCBA} = \mathrm{X}, \quad \ldots \quad (2)$$

with the proviso that α_2, β_2,. ξ_2 are then the horizontal and ${}_1$, $\beta_1\ldots$ ξ_1 the vertical nullities of A, B,...X in (2) ; for we can regard (2) as the conjugate of (1).

When the inverted standard forms are regarded as fundamental, all the results obtained for the product (1) can be converted into the corresponding results for the product

$$\boxed{a}^{\,p}_{\,m}\ \boxed{b}^{\,q}_{\,p}\ \boxed{c}^{\,r}_{\,q}\ \boxed{d}^{\,n}_{\,r} = \boxed{x}^{\,n}_{\,m} \quad \text{or} \quad \mathrm{ABCD} = \mathrm{X} \quad \ldots \quad (3)$$

by replacing

$$m, p, q, r, n\ ;\ i, u, v, w, j\ ;\ \alpha_1, \alpha_2\ ;\ \alpha_{10}. \alpha_{20}\ \ldots$$

by $$n, r, q, p, m\ ;\ j, w, v, u, i\ ;\ \alpha_2, \alpha_1\ ;\ \alpha_{20}, \alpha_{10}\ ;\ \ldots$$

Then every letter has the same meaning for the product (3) as it had for the product (1), α_1, β_1,...ξ_1 being horizontal nullities, and α_1, β_2...ξ_2 being vertical nullities. Thus the general formula for a parameter x_{ji} of X becomes

$$x_{ji} = \Sigma a_{ui}\ b_{vu}\ c_{wv}\ d_{jw} = \Sigma d_{jw}\ c_{wv}\ b_{vu}\ a_{ui}\ (i-j=k),$$

the possible values of i, j and k, and the range of the summation being as before. The only difference is that in passing from (1) to (3) the words 'initial' and 'final' must be interchanged in speaking of the horizontal and vertical parametric rows.

A note on the construction by points of a trinodal quartic.

By

Girindralal Mukherjee

[*Read February 29th, 1920*]

1. In this note I propose to discuss a solution of the problem :—

To construct by *points* a trinodal quartic, of which the three nodes and five other points are given.

The solution is based upon two theorems which may be stated as follows :—

I. A, B, C are three given points and S is a given conic, Through one of these points, say A, two lines AP and AX are drawn meeting BC in P and X respectively and making with AB and AC a pencil of constant anharmonic ratio such that A (B, C, P, X)$=k$. If now from the foot of one of these lines (say X of the line AX) two tangents are drawn to the conic S, they will meet the other line AP in a pair of points P′, P″ which will trace out a trinodal quartic, having A, B, P for nodes and touching S at four points.

II. A, B, C are the three nodes of a trinodal quartic U. Through one of the nodes, say A, two lines AP and AX are drawn, meeting BC in P and X respectively and making with AB, AC a pencil of constant anharmonic ratio such that A (B, C, P, X)$=k$. If now one of these lines, say AP, meet the curve U in the two points P,′ P″ ; then the lines joining X (the foot of the other line AX) to these points will envelop a conic S, having quartic contact with U.

The following few points may be noted in connection with these theorems :—

(*a*) The lines AB and AX trace out two pencils which are concentric and homographic, having AB, AC, for their common self-corresponding elements.

The pencil traced out by AP will in future be referred to as the *first* pencil ; and that traced out by AX as the *second* pencil.

(*b*) Let U be the quartic traced out by the points P′, P″ in theorem I. It can be easily seen that if BQ and BY be drawn through B meeting AC in Q and Y respectively and making with BC and BA a pencil of which the anharmonic ratio is $1-\frac{1}{k}$; that is to say $B(C, A, Q, Y)=1-\frac{1}{k}$, then BQ will meet the tangents drawn to S from Y in a pair of points which will lie on U. Similarly if through C, CR and CZ be drawn meeting AB in R and Z respectively such that $C\ (A, B, R, Z)=\frac{1}{1-k}$; then CR will meet the tangents to S from Z in a pair of points lying on U.

2. From what has been said above it is clear that the first step in the solution of the problem under consideration will be to determine the conic S. Now the conic S can be determined in the following manner :—

Let A, B, C be the three nodes and P_1, P_2, P_3, P_4, P_5 be the five other given points. Join one of the points A, B, C (say A) to each of the five given points P. Drawn AX_1, AX_2, AX_3, AX_1, AX_5 meeting BC in X_1, X_2, X_3, X_4, X_5 respectively such that

$$A\ (B, C, P_1, X_1)=A\ (B, C, P_2, X_2)=\ldots=k.$$

Drawn a conic touching the five lines P_1X_1, $P\ X_2$, P_3X_3, P_4X_4, P_5X_5; this conic will be the required conic S. Having determined S, the next step will be to determine the points where lines drawn through either A, B or C meet the tangents drawn to this conic S from the feet of the corresponding lines as explained above. These points will be points on the quartic U to be drawn.

It is easily seen that in solving the problem we require the help of the ruler and compasses only. Hence the problem of constructing a trinodal quartic of which the three nodes and five other points are given may be regarded as a problem of the second degree.

3. I shall show that the curve U generated above is a trinodal quartic curve, having A, B, C for nodes; or in other words that it possesses all the following properties of a trinodal quartic, *viz.*—

(1) that the order of the curve is four ;

(2) that it has got three nodes and A, B, C are these nodes ;

(3) that it has *four* double tangents ;

(4) that it has *six* inflexional points ;

(5) that the class of the curve is six,

(1) The degree of a curve can be ascertained by determining the number of its intersections with a right line. Let L be any right line and it is required to determine the number of its intersections with U. Let L meet BC in O and AB, AC, AP_1, AP_2, AP_3, AP_4, AP_5 in B′, C′, P'_1, P'_2, P'_3, P'_4, P'_5 respectively; then it is easily seen that the two ranges of points O (B, C, X_1, X_2, X_3, X_4, X_5) and O (B′, C′, P'_1, P'_2, P'_3, P'_4, P'_5) are homographic but not in perspective. Hence the lines BB′, CC′, P'_1X_1, P'_2X_2, etc., joining corresponding points of the two hamographic ranges will envelop a conic. Let C be this conic; it will evidently also touch the bases L and BC. Now, if L_1 be a point common to L and U and M_1 be the point on BC such that A (B, C, L_1, M_1)=A (B, C, P'_1, X_1) $=k$; then L_1M_1 will touch both C and S. Hence L_1 to be a common points of intersection of L and U must also lie on a common tengent of C and S. As there are four common tangents to C and S; the number of points common to L and U must also be *four* and these points are the four points of intersection of L with the four common tangents of C and S. Thus it is proved that the degree of the curve is four.

(2) That A, B, C are double points on U can be easily seen by considering the intersections with U of the lines AB, AC, BC.

Again let any line AP through A meet U in the points P′, P″ and let X be a point on BC, such that

$$A\ (B,\ C,\ P',\ X)=A\ (B,\ C,\ P_1,\ X_1)=k$$

then XP′, XP″ will be tangents to S. If now AP′P″ be made to rotate round A until P′ moves upto and ultimately coincides with A; then AP″ will be a tangent to the curve at A. But then XP′ becomes coincident with AX, one of the tangents from A to S. Thus a tangent to the curve U at A is a ray of the first pencil corresponding to a tangent to S from A, regarded as a ray of the second pencil. As there can be drawn two tangents to S from A, it at once follows that there can be drawn two tangents to the curve U at A, and they are the rays of the first pencil corresponding to the tangents to S from A, considered as rays of the second pencil.

Similarly also it can be shown that two tangents can be drawn to U at each of the points B and C and they are lines corresponding to the tangents drawn from them respectively to S.

In this way also it is seen that A, B, C are double points on the curve U.

The cases when these tangents will be real, imaginary or coincident can be easily distinguished.

(3) If in (1) L be a double tangent to the curve, C will have double contact with S. Hence the number of double tangents to the curve is evidently equivalent to the number of such conics C that can be drawn so as to touch the three given lines BC, CA, AB and have double contact with the given conic S. The number of such conics, as we know, is four and consequantly the number of bitangents to the curve is *four*.

Let S intersect the sides BC, CA, AB of the triangle ABC in the three pairs of points A_1, A_2, B_1, B_2 ; C_1, C_2 respectively in the order in which they are written ; then it can be shown that the four Pascal lines

$$\begin{array}{ccc} C_2A_2B_2C_1A_1B_1 & C_2A_1B_2C_1A_2B_1 & C_2A_1B_1C_1A_2B_2 \\ | & | & | \\ \overbrace{C_2A_2\cdot C_1A_1,} & \overbrace{C_2A_1\cdot C_1A_2,} & \overbrace{C_2A_1\cdot C_1A_2,} \\ A_2B_2\cdot A_1B_1, & A_1B_2\cdot A_2B_1, & A_1B_1\cdot A_2B_2, \\ B_2C_1\cdot B_1C_2. & B_2C_1\cdot B_1C_2, & B_1C_1\cdot B_2C_2, \end{array}$$

$$\begin{array}{c} C_1A_1B_2C_2A_2B_1 \\ | \\ \overbrace{C_1A_1\cdot C_2A_2,} \\ A_1B_2\cdot A_2B_1, \\ B_2C_2\cdot B_1C_1. \end{array}$$

(where the Pascal line $C_2A_2B_2C_1A_1B_1$ is obtained by joining the three points of intersection of the three pairs of lines C_2A_2, C_1A_1 ; A_2B_2, A_1B_1 ; B_2C_1, B_1C_2 respectively) are the four common chords of contact of the conics C and S.

(4) If L be an inflexional tangent to the curve U at any point L_1 ; then C will have contact of the second order with S. Hence the number of inflexional tangents, and consequently of inflexional points on the curve, will be the same as the number of conics C that

can be drawn touching three given lines and osculating a given conic. As the latter number is six, it follows that the number of *points of inflexion* on the curve is at most *six*.

(5) If L touches the curve U at L_1; then C and S will touch one another; and if M_1 be the point on BC such that

$$A\ (B, C, L_1, M_1) = A\ (B, C, P_1, X_1) = k$$

then L_1 M_1 will touch both C and S at their common points of contact. Therefore L will be the second tangent drawn from L_1 to C.

If L passes through a point P which is not on the curve U and M be a point on BC such that

$$A\ (B, C, P, M) = A\ (B, C, P_1, X_1) = k$$

then PM will touch C. Hence when P is given, to determine PL_1 we have to draw the conic C of which it is a conic touching four given lines (BC, CA, AB, PM) and a given conic S. As the problem of determining C admits of six different solutions, it follows that *six* tangents can be drawn to the curve U from P. In other words the *class* of the curve is *six*.

When P is on the curve, PM touches S also; and in this case two of the six solutions become identical and this double solution will then correspond to the tangent to the curve at P. Thus when P is on the curve, besides the tangent at P, there can be drawn to the curve four other tangents from it.

If P coincides with any one of the points A, B or C, this construction will fail; and in that case it can be shown that only two tangents can be drawn to the curve from each of these points. By making AP in sub-sec. (2) rotate round A until P'' moves up to and coincides with P', it can be shown that the tangents from A to the curve will be rays of the *first* pencil corresponding to the lines AA_1, AA_2 considered as rays of the *second* pencil (A_1, A_2 being the points of intersection of BC with S).

Similarly the tangents from B and C can be determined.

4. I have shown that the curve U possesses all the Pluckerian characteristics of a trinodal quartic. It now remains to show that the curve thus generated is a *unique* one; for it may be argued that by varying k, different curves satisfying the given conditions can be drawn. But upon examination this will be found not to be the case;

for if it were possible to draw two different curves by giving two different values to λ, then we would have the absurdity of two quartic curves intersecting in $(3\times 4+5)=$ seventeen or more than sixteen, points.

5. I conclude this note by giving a few additional problems on the trinodal quartic, the solutions of which can be made to depend upon the two theorems stated in Art. 1. The problems with their solutions are given below :—

I. Given the three nodes A, B, C ; one other point P and the conic S ; construct the quartic.

Here the first thing for determination is the constant of anharmonic ratio (k). If the two tangents from P to S meet BC in X_1 and X_2 ; then k will be determined either by $A\ (B, C, P, X_1)$ or $A\ (B, C, P, X_2)$. Hence there will be two solutions.

II. Given the three other poinis P_1, P_2, P_3 and a bitangent L : to construct the quartic.

Here the conic S is to be first determined. This can be done in the following manner :—Take the points X_1, X_2, X_3 on BC such that $A\ (B, C, P_1, X_1)=A\ (B, C, P_2, X_2)=A\ (B, C, P_3, X_3)$. Let AP_1, AP_2, AP_3, meet L in P'_1, P'_2, P'_3 respectively. Draw a conic C touching BC, CA, AB, L and $P'_1X'_1$, which will also touch P'_2X_2, P'_3X_3. Draw another conic (S) touching the lines P_1X_1, P_2X_2, P_3X_3 and having double contact with C. This conic will be the required conic S.

This problem admits of four different solutions.

III. Given the three double points A, B, C ; one other point P and two bitangents L_1 and L_2 : to construct the quartic.

Here the conic S can be determined in the following manner :—

Take any point X on BC ; and let P_1, P_2 be the points where AP meet L_1, L_2 respectively. Draw a conic C_1 touching BC, CA, AB, L_1, P_1X and another conic C_2 touching BC, CA, AB, L_2 and P_2X. Draw a conic (S) touching PX and having double contant with both C_1 and C_2. This last-drawn conic will be the required conic S.

This problem also, like the preceding two admits of more than one solution.

On Liquid Motion inside Rotating Arcs of Three and Four Confocal Parabolas

By

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The problem of determining the two-dimensional motion of a homogeneous incompressible liquid inside rotating cylinders has engaged the attention of many eminent mathematicians, but up to now the problem has been solved for only a small number of cases. The present state of analysis does not enable us to attack with success the general problem for any arbitrary boundary. A solution for the case in which the cross-section of the rotating cylinder consists of arcs of two confocal parabolas has been given by Ferrers.[1] The object of the present paper is to present the solution for two new cases, *viz.*, (1) when the cross-section consists of *three confocal parabolas* and (2) when the cross-section consists of *four confocal parabolas*, all belonging to the same orthogonal system.

It may be mentioned that the hydrodynamical solutions given in this paper also furnish the solutions of the torsion problem in the Theory of Elasticity for the same boundaries.

§ 1. *Preliminary Observations.*

Take the common focus of the parabolic arcs of a cross-section as the origin and the common axis as the axis of x. If λ and μ be the parameters of the two parabolas which pass through the point (x, y), we have

$$x=\lambda-\mu,$$

$$y^2=4\lambda\mu,$$

λ being the parameter of the parabola whose axis is the negative x-axis.

Let $\lambda=\xi^2$ and $\mu=\eta^2$,

so that we have

$$\left.\begin{aligned} x&=\xi^2-\eta^2 \\ y&=\pm 2\xi\eta \end{aligned}\right\} \qquad \ldots \quad (1)$$

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[1] Ferrers: "On the motion of water contained in certain cylindrical vessels, &c.," *Quarterly Journal*, Vol. 17.

To avoid ambiguity we shall suppose that η is *always positive* while ξ may have *either sign*, and instead of (1) we shall write

$$\left.\begin{aligned} x&=\xi^2-\eta^2, \\ y&=2\xi\eta. \end{aligned}\right\} \qquad \dots \quad (2)$$

Thus η=constant are complete parabolas and ξ=constant are half-parabolas, the upper halves corresponding to positive values and the lower halves to negative ones.

It is easily seen from (2) that ξ and η are conjugate functions given by the relation

$$x+iy=(\xi+i\eta)^2.$$

Hence the stream function ψ must satisfy the condition

$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}=0 \qquad \dots \quad (3)$$

at all points occupied by the liquid.

Also, if the cylinder rotates with angular velocity ω about an axis through the focus, we must have

$$\psi=\tfrac{1}{2}\omega(x^2+y^2)+\text{constant} \qquad \dots \quad (4)$$

at all points on the boundary.

Now from (2) we have

$$x^2+y^2=(\xi^2+\eta^2)^2=(\xi^4-6\xi^2\eta^2+\eta^4)+8\xi^2\eta^2.$$

Since $\xi^4-6\xi^2\eta^2+\eta^4$ satisfies the equation (3), we shall assume

$$\psi=\tfrac{1}{2}\omega(\xi^4-6\xi^2\eta^2+\eta^4)+\psi_1.$$

Then ψ_1 will have to satisfy the conditions

$$\frac{\partial^2\psi_1}{\partial\xi^2}+\frac{\partial^2\psi_1}{\partial\eta^2}=0 \qquad \dots \quad (5)$$

at all points occupied by the liquid, and

$$\psi_1=4\omega\xi^2\eta^3+\text{constant} \qquad \dots \quad (6)$$

at all points of the boundary.

§ 2. *Three Confocal Parabolas.*

Let the cylinder be bounded by the three confocal parabolic cylinders $\eta=\eta_1$, $\eta=\eta_2$ and $\xi=\pm\xi_1$.

Let $\eta_2>\eta_1$, $\eta_2-\eta_1=2\beta$ and $\eta_2+\eta_1=2\delta$, so that $\eta_2=\delta+\beta$, and $\eta_1=\delta-\beta$.

Assume

$$\psi_1 = \sum_{0}^{\infty} \cos(2n+1)\frac{\pi\xi}{2\xi_1}\left[A_n \sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1) + B_n \sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)\right]$$

$$+ \sum_{0}^{\infty} C_n \cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\cosh(2n+1)\frac{\pi\xi}{2\beta}.$$

The equation (5) is obviously satisfied. The equation (6) will be satisfied if

$\psi_1 = 4\omega\xi^2\eta_1^2$ when $\eta=\eta_1$ and $-\xi_1<\xi<\xi_1$

$\psi_1 = 4\omega\xi^2\eta_2^2$ when $\eta=\eta_2$ and $-\xi_1<\xi<\xi_1$,

and $\psi_1 = 4\omega\xi_1^2\eta^2$ when $\xi=\pm\xi_1$ and $\eta_1<\eta<\eta_2$.

Hence, putting $\eta=\eta_2$, we must have, for all values of ξ between $-\xi_1$ and ξ_1,

$$4\omega\xi^2\eta_2^2 = \sum_{0}^{\infty} A_n \cos(2n+1)\frac{\pi\xi}{2\xi_1}\sinh\frac{2n+1)\pi\beta}{\xi_1}.$$

Multiplying both sides by $\cos(2n+1)\frac{\pi\xi}{2\xi_1}d\xi$ and integrating between the limits $-\xi_1$ and ξ_1, we get

$$\xi_1 A_n \sinh\frac{(2n+1)\pi\beta}{\xi_1} = 4\omega\eta_2^2\int_{-\xi_1}^{\xi_1}\xi^2\cos(2n+1)\frac{\pi\xi}{2\xi_1}d\xi$$

$$= \frac{64\omega\xi_1^3\eta_2^2}{\pi^3}\int_0^{\frac{\pi}{2}} z^2\cos(2n+1)z\,dz$$

$$= \frac{(-1)^n\,64\omega\xi_1^3\eta_2^2}{(2n+1)\pi}\left[\frac{1}{4}-\frac{2}{(2n+1)^2\pi^2}\right]$$

$$\therefore A_n = \frac{(-1)^n}{(2n+1)\pi}\cdot\frac{64\omega\xi_1^2\eta_2^2\left[\frac{1}{4}-\frac{2}{(2n+1)^2\pi^2}\right]}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}.$$

Similarly

$$B_n = \frac{(-1)^n}{(2n+1)\pi} \cdot \frac{64\omega\xi_1^2\eta_1^2\left[\frac{1}{4} - \frac{2}{(2n+1)^2\pi^2}\right]}{\sinh\frac{(2n+1)\pi\beta}{\xi_2}}.$$

To find C_n put $\xi = \pm\xi_1$. Then we must have, for all values of η between η_1 and η_2,

$$4\omega\xi_1^2\eta^2 = \sum_0^\infty C_n \cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\cosh\frac{(2n+1)\pi\xi_1}{2\beta}.$$

Multiplying both sides by $\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)d\eta$ and integrating between the limits η_1 and η_2, we obtain

$$\beta C_n \cosh\frac{(2n+1)\pi\xi_1}{2\beta} = 4\omega\xi_1^2\int_{\eta_1}^{\eta_2}\eta^2\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)d\eta$$

$$= 4\omega\xi_1^2\int_{-\beta}^{\beta}(z^2+2z\delta+\delta^2)\cos\frac{(2n+1)\pi z}{2\beta}dz.$$

Now $$\int_{-\beta}^{\beta} z^2\cos\frac{(2n+1)\pi z}{2\beta}dz = \frac{(-1)^n 4\beta^3}{(2n+1)\pi}\left[1 - \frac{8}{(2n+1)^2\pi^2}\right]$$

$$\int_{-\beta}^{\beta} z\cos\frac{(2n+1)\pi z}{2\beta}dz = 0,$$

$$\int_{-\beta}^{\beta}\cos\frac{(2n+1)\pi z}{2\beta}dz = \frac{(-1)^n\cdot 4\beta}{(2n+1)\pi}.$$

Thus $$C_n = \frac{(-1)^n 16\omega\xi_1^2}{(2n+1)\pi} \cdot \frac{\left[(\beta^2+\delta^2) - \frac{8\beta^2}{(2n+1)^2\pi^2}\right]}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}.$$

We thus have

$$\psi=\tfrac{1}{2}\omega(\xi^4-6\xi^2\eta^2+\eta^4)+\psi_1,$$

where

$$\psi_1=16\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\}\cos\frac{(2n+1)\pi\xi}{2\xi_1}$$

$$\times\left[\frac{\eta_2^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)+\eta_1^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}\right]$$

$$+16\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\left\{(\beta^2+\delta^2)-\frac{8\beta^2}{(2n+1)^2\pi^2}\right\}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)$$

$$\times\frac{\cosh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}$$

$$=16\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\left[\frac{\eta_2^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)+\eta_1^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}\right.$$

$$\left.\times\cos\frac{(2n+1)\pi\xi}{2\xi_1}+(\beta^2+\delta^2)\frac{\cosh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\right]$$

$$-128\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}\left[\frac{\eta_2^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)+\eta_1^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}\right.$$

$$\left.\times\cos\frac{(2n+1)\pi\xi}{2\xi_1}+\beta^2\frac{\cosh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\right].$$

Remembering that $\phi+i\psi$ is a function of $\xi+i\eta$, the expression for the velocity potential ϕ can be readily worked out.

We thus obtain

$$\phi=-2\omega\xi\eta(\xi^2-\eta^2)+\phi_1,$$

where

$$\phi_1=16\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\left[\frac{\eta_2^2\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)+\eta_1^2\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}\right.$$

$$\left.\times\sin\frac{(2n+1)\pi\xi}{2\xi_1}-(\beta^2+\delta^2)\frac{\sinh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\right]$$

$$-128\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}\left[\frac{\eta_2^2\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)+\eta_1^2\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}\right.$$

$$\left.\times\sin\frac{(2n+1)\pi\xi}{2\xi_1}-\beta^2\frac{\sinh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\right].$$

§ 3. *Simplified Results.*

The results obtained in the preceding article can be simplified to a certain extent in the following manner.

Substituting the values of η_1 and η_2 in terms of β and δ and observing that

$$\sinh\frac{(2n+)\pi}{2\xi_1}(\eta-\eta_1)+\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)$$

$$=2\sinh\frac{(2n+1)\pi\beta}{2\xi_1}\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta),$$

$$\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)-\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)$$

$$=2\cosh\frac{(2n+1)\pi\beta}{2\xi_1}\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta),$$

and $$\sinh\frac{(2n+1)\pi\beta}{\xi_1}=2\sinh\frac{(2n+1)\pi\beta}{2\xi_1}\cosh\frac{(2n+1)\pi\beta}{2\xi_1},$$

we find that ψ_1 can be written as

$$\psi_1=16\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\{(\beta^2+\delta^2)(P+Q)+2\beta\delta R\}$$

$$-128\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}\{(\beta^2+\delta^2)P+\beta^2Q+2\beta\delta R\},\quad\dots\quad(7)$$

where $$P=\frac{\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta)}{\cosh\frac{(2n+1)\pi\beta}{2\xi_1}}\cos\frac{(2n+1)\pi\xi}{2\xi_1},\quad\dots\quad(8)$$

$$Q=\frac{\cosh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta),\quad\dots\quad(9)$$

and $$R=\frac{\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta)}{\sinh\frac{(2n+1)\pi\beta}{2\xi_1}}\cos\frac{(2n+1)\pi\xi}{2\xi_1}.\quad\dots\quad(10)$$

Now it is obvious that, when $\eta=\eta_1$ and $-\xi_1<\xi<\xi_1$,

$$\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}(P+Q)=\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\cos\frac{(2n+1)\pi\xi}{2\xi_1}=\tfrac{1}{4}.$$

In a similar manner it will be found that, when $\eta=\eta_2$ and $-\xi_1<\xi<\xi_1$ or when $\xi=\pm\xi_1$ and $\eta_1<\eta<\eta_2$,

$$\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}(P+Q)=\tfrac{1}{4}.$$

Thus the above equation holds for all points on the boundary.

Also $$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\left\{\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}(P+Q)\right\}=0,$$

Hence, everywhere within the cylinder we must have

$$\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)\pi}(\mathrm{P}+\mathrm{Q})=\tfrac{1}{4}. \qquad \ldots \quad (11)$$

Integrating the equation (11) with respect to η, we have

$$\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^2\pi^2}\left\{2\xi_1\frac{\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta)}{\cosh\frac{(2n+1)\pi\beta}{2\xi_1}}\cos\frac{(2n+1)\pi\xi}{2\xi_1}\right.$$

$$\left.+2\beta\frac{\cosh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\right\}=\frac{\eta}{4}+f(\xi).$$

To determine $f(\xi)$ put $\eta=\delta$ in both sides. Then the lefthand side vanishes.

Thus $\qquad f(\xi)=-\dfrac{\delta}{4}.$

Substituting this value and integrating again with respect to η, we obtain

$$\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^3\pi^3}\left\{4\xi_1^2\mathrm{P}-4\beta^2\mathrm{Q}\right\}=\frac{(\eta-\delta)^2}{8}+\mathrm{F}(\xi). \qquad \ldots \quad (12)$$

Since the lefthand side remains unchanged when $-\xi$ is written for ξ, $\mathrm{F}(\xi)$ must be an even function of ξ.

To determine this function operate on both sides by $\dfrac{\partial^2}{\partial\xi^2}+\dfrac{\partial^2}{\partial\eta^2}$. Then the lefthand side vanishes and we have

$$\mathrm{F}''(\xi)=-\tfrac{1}{4}.$$

$$\therefore \quad \mathrm{F}(\xi)=-\tfrac{1}{8}\xi^2+k,$$

where k is a constant of integration.

To find the value of k substitute this value of $\mathrm{F}(\xi)$ in (12) and put $\xi=\pm\xi_1$ and $\eta=\eta_1$ or η_2. Then the lefthand side disappears and we have

$$0=\tfrac{1}{8}(\beta^2-\xi_1^2)+k.$$

Thus $$k=\tfrac{1}{8}(\xi_1^2-\beta^2)$$

and $$\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}(\xi_1^2 P-\beta^2 Q)=\tfrac{1}{32}\{(\eta-\delta)^2-\xi^2+\xi_1^2-\beta^2\} \quad \ldots \quad (13)$$

Let $$\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}\{(\beta^2+\delta^2)P+\beta^2 Q\}=U. \quad \ldots \quad (14)$$

From (13) and (14) we easily obtain

$$U=-\tfrac{1}{32}\{(\eta-\delta)^2-\xi^2+\xi_1^2-\beta^2\}+(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}P, \quad (15)$$

or

$$U=\frac{(\beta^2+\delta^2)}{32\xi_1^2}\{(\eta-\delta)^2-\xi^2+\xi_1^2-\beta^2\}$$
$$+\frac{\beta^2}{\xi_1^2}(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}Q. \quad \ldots \quad (16)$$

From (7), (11) and (14) we have

$$\psi_1=4\omega\xi_1^2(\beta^2+\delta^2)-128\omega\xi_1^2U+32\beta\delta\omega\xi_1^2\sum_0^\infty \frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\}R.$$

Using (15) or (16) we finally have

$$\psi_1=4\omega\xi_1^2\{(\eta-\delta)^2-\xi^2+\xi_1^2+\delta^2\}-128\omega\xi_1^2(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}P$$
$$+32\beta\delta\omega\xi_1^2\sum_0^\infty \frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\}R,$$

or

$$\psi_1=4\omega(\beta^2+\delta^2)\{\xi^2-(\eta-\delta)^2+\beta^2\}-128\omega\beta^2(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}Q$$
$$+32\beta\delta\omega\xi_1^2\sum_0^\infty \frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\}R,$$

where P. Q and R are given by the relations (8), (9) and (10).

The corresponding expressions for ϕ_1 may be easily deduced.

Thus

$$\phi_1=8\omega\xi_1^2\xi(\eta-\delta)-128\omega\xi_1^2(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}P'$$
$$+32\beta\delta\omega\xi_1^2\sum_0^\infty \frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\}R',$$

or

$$\phi_1 = -8\omega(\beta^2+\delta^2)\xi(\eta-\delta) + 128\omega\beta^2(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3} Q'$$

$$+32\beta\delta\omega\xi_1^2 \sum_0^\infty \frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\} R',$$

where $$P' = \frac{\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta)}{\cosh\frac{(2n+1)\pi\beta}{2\xi_1}} \sin\frac{(2n+1)\pi\xi}{2\xi_1},$$

$$Q' = \frac{\sinh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}} \sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta).$$

and $$R' = \frac{\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta)}{\sinh\frac{(2n+1)\pi\beta}{2\xi_1}} \sin\frac{(2n+1)\pi\xi}{2\xi_1}.$$

§ 4. *Four Confocal Parabolas.*

Suppose the cylinder is bounded by the four confocal parabolic cylinders $\xi=\xi_1$, $\xi=\xi_2$, $\eta=\eta_1$ and $\eta=\eta_2$

Let $\xi_2>\xi_1.\ \eta_2>\eta_1:$

$\xi_2-\xi_1=2\alpha,\ \eta_2-\eta_1=2\beta$

and $\xi_2+\xi_1=2\gamma,\ \eta_2+\eta_1=2\delta,$

so that $\xi_2=\gamma+\alpha,\ \xi_1=\gamma-\alpha$

and $\eta_2=\delta+\beta,\ \eta_1=\delta-\beta.$

Assume

$$\psi_1 = \sum_0^\infty \cos\frac{(2n+1)\pi}{2\alpha}(\xi-\gamma)\left[A_n\sinh\frac{(2n+1)\pi}{2\alpha}(\eta-\eta_1) + B_n\sinh\frac{(2n+1)\pi}{2\alpha}(\eta_2-\eta)\right]$$

$$+\sum_0^\infty \cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\left[C_n\sinh\frac{(2n+1)\pi}{2\beta}(\xi-\xi_1) + D_n\sinh\frac{(2n+1)\pi}{2\beta}(\xi_2-\xi)\right].$$

Then proceeding exactly as in the preceding articles we obtain the following results.

$$A_n = 16\omega\eta_2^2 \cdot \frac{\dfrac{(-1)^n}{(2n+1)\pi}\left[(\alpha^2+\gamma^2) - \dfrac{8\alpha^2}{(2n+1)^2\pi^2} \right]}{\sinh \dfrac{(2n+1)\pi\beta}{\alpha}},$$

$$C_n = 16\omega\xi_2^2 \cdot \frac{\dfrac{(-1)^n}{(2n+1)\pi}\left[(\beta^2+\delta^2) - \dfrac{8\beta^2}{(2n+1)^2\pi^2} \right]}{\sinh \dfrac{(2n+1)\pi\alpha}{\beta}},$$

B_n and D_n are what A_n and C_n become when η_1^2 and ξ_1^2 are written for η_2^2 and ξ_2^2 respectively;

whence

$$\psi_1 = 16\omega\sum_0^\infty \frac{(-1)^n}{(2n+1)\pi}\left[(\alpha^2+\gamma^2) \cos\frac{(2n+1)\pi}{2\alpha}(\xi-\gamma)\right.$$

$$\times \frac{\eta_2^2 \sinh\dfrac{(2n+1)\pi}{2\alpha}(\eta-\eta_1) + \eta_1^2 \sinh\dfrac{(2n+1)\pi}{2\alpha}(\eta_2-\eta)}{\sinh \dfrac{(2n+1)\pi\beta}{\alpha}}$$

$$+(\beta^2+\delta^2) \cos \frac{(2n+1)\pi}{2\beta}(\eta-\delta)$$

$$\left.\times \frac{\xi_2^2 \sinh \dfrac{(2n+1)\pi}{2\beta}(\xi-\xi_1) + \xi_1^2 \sinh\dfrac{(2n+1)\pi}{2\beta}(\xi_2-\xi)}{\sinh \dfrac{(2n+1)\pi\alpha}{\beta}}\right]$$

$$-128\omega\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}\left[\alpha^2 \cos \frac{(2n+1)\pi}{2\alpha}(\xi-\gamma)\right.$$

$$\times \frac{\eta_2^2 \sinh\dfrac{(2n+1)\pi}{2\alpha}(\eta-\eta_1) + \eta_1^2 \sinh\dfrac{(2n+1)\pi}{2\alpha}(\eta_2-\eta)}{\sinh\dfrac{(2n+1)\pi\beta}{\alpha}}$$

$$+\beta^2 \cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)$$

$$\times\frac{\xi_2^2 \sinh\frac{(2n+1)\pi}{2\beta}(\xi-\xi_1)+\xi_1^2 \sinh\frac{(2n+1)\pi}{2\beta}(\xi_2-\xi)}{\sinh\frac{(2n+1)\pi\alpha}{\beta}}\Bigg]$$

and

$$\phi_1=16\omega\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)\pi}\Bigg[(\alpha^2+\gamma^2)\sin\frac{(2n+1)\pi}{2\alpha}(\xi-\gamma)\;\frac{\eta_2^2\text{Y}_1+\eta_1^2\text{Y}_2}{\sinh\frac{(2n+1)\pi\beta}{\alpha}}$$

$$-(\beta^2+\delta^2)\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\;\frac{\xi_2^2\text{X}_1+\xi_1^2\text{X}_2}{\sinh\frac{(2n+1)\pi\alpha}{\beta}}\Bigg]$$

$$-128\omega\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^3\pi^3}\Bigg[\alpha^2\sin\frac{(2n+1)\pi}{2\alpha}(\xi-\gamma)\;\frac{\eta_2^2\text{Y}_1+\eta_1^2\text{Y}_2}{\sinh\frac{(2n+1)\pi\beta}{\alpha}}$$

$$-\beta^2\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\;\frac{\xi_2^2\text{X}_1+\xi_1^2\text{X}_2}{\sinh\frac{(2n+1)\pi\alpha}{\beta}}\Bigg],$$

where $\text{X}_1=\cosh\frac{(2n+1)\pi}{2\beta}(\xi-\xi_1),$

$\text{X}_2=\cosh\frac{(2n+1)\pi}{2\beta}(\xi_2-\xi),$

$\text{Y}_1=\cosh\frac{(2n+1)\pi}{2\alpha}(\eta-\eta_1),$

and $\text{Y}_2=\cosh\frac{(2n+1)\pi}{2\alpha}(\eta_2-\eta)$

Substituting the values of ξ_1, ξ_2, η_1 and η_2 in terms of α, β, γ and δ and adopting the method of the preceding articles, we get the following additional results :

$$\psi_1=16\omega\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)\pi}\big[(\alpha^2+\gamma^2)(\beta^2+\delta^2)(\text{P}+\text{Q})+2\beta\delta(\alpha^2+\gamma^2)\text{R}$$

$$+2\alpha\gamma(\beta^2+\delta^2)\text{S}\big]$$

$$-128\omega\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^3\pi^3}\big[\alpha^2(\beta^2+\delta^2)\text{P}+\beta^2(\alpha^2+\gamma^2)\text{Q}+2\alpha^2\beta\delta\text{R}+2\beta^2\alpha\gamma\text{S}\big],$$

where

$$P=\frac{\cosh\frac{(2n+1)\pi}{2\alpha}(\eta-\delta)}{\cosh\frac{(2n+1)\pi\beta}{2\alpha}}\cos\frac{(2n+1)\pi}{2\alpha}(\xi-\gamma), \quad \ldots \quad (17)$$

$$Q=\frac{\cosh\frac{(2n+1)\pi}{2\beta}(\xi-\gamma)}{\cosh\frac{(2n+1)\pi\alpha}{2\beta}}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta), \quad \ldots \quad (18)$$

$$R=\frac{\sinh\frac{(2n+1)\pi}{2\alpha}(\eta-\delta)}{\sinh\frac{(2n+1)\pi\beta}{2\alpha}}\cos\frac{(2n+1)\pi}{2\alpha}(\xi-\gamma), \quad \ldots \quad (19)$$

and

$$S=\frac{\sinh\frac{(2n+1)\pi}{2\beta}(\xi-\gamma)}{\sinh\frac{(2n+1)\pi\alpha}{2\beta}}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta). \quad \ldots \quad (20)$$

We shall also find that, at all points on and within the boundary,

$$\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)\pi}(P+Q)=\tfrac{1}{4};$$

$$\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^3\pi^3}[\alpha^2P-\beta^2Q]=\tfrac{1}{32}[(\eta-\delta)^2-(\xi-\gamma)^2+\alpha^2-\beta^2],$$

and if

$$\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^3\pi^3}[\alpha^2(\beta^2+\delta^2)P+\beta^2(\alpha^2+\gamma^2)Q]=U,$$

then

$$U=-\frac{\alpha^2+\gamma^2}{32}[(\eta-\delta)^2-(\xi-\gamma)^2+\alpha^2-\beta^2]$$

$$+\alpha^2(\alpha^2+\beta^2+\gamma^2+\delta^2)\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^3\pi^3}P;$$

or

$$U=\frac{\beta^2+\delta^2}{32}[(\eta-\delta)^2-(\xi-\gamma)^2+\alpha^2-\beta^2]$$

$$+\beta^2(\alpha^2+\beta^2+\gamma^2+\delta^2)\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}Q.$$

Finally

$$\psi_1=4\omega(\alpha^2+\gamma^2)[(\eta-\delta)^2-(\xi-\gamma)^2+\alpha^2+\delta^2]$$

$$+32\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}[\beta\delta(\alpha^2+\gamma^2)R+\alpha\gamma(\beta^2+\delta^2)S]$$

$$-128\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}[\alpha^2(\alpha^2+\beta^2+\gamma^2+\delta^2)P+2\alpha^2\beta\delta R+2\beta^2\alpha\gamma S],$$

or
$$\psi_1=-4\omega(\beta^2+\delta^2)[(\eta-\delta)^2-(\xi-\gamma)^2-\beta^2-\gamma^2]$$

$$+32\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}[\beta\delta(\alpha^2+\gamma^2)R+\alpha\gamma(\beta^2+\delta^2)S]$$

$$-128\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}[\beta^2(\alpha^2+\beta^2+\gamma^2+\delta^2)Q+2\alpha^2\beta\delta R+2\beta^2\alpha\gamma S],$$

where P, Q, R and S are given by (17), (18), (19) and (20),

and
$$\phi_1=8\omega(\alpha^2+\gamma^2)(\xi\eta-\delta\xi-\gamma\eta)$$

$$+32\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}[\beta\delta(\alpha^2+\gamma^2)R'-\alpha\gamma(\beta^2+\delta^2)S']$$

$$-128\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}[\alpha^2(\alpha^2+\beta^2+\gamma^2+\delta^2)P'+2\alpha^2\beta\delta R'-2\beta^2\alpha\gamma S'],$$

or
$$\phi_1=-8\omega(\beta^2+\delta^2)(\xi\eta-\delta\xi-\gamma\eta)$$

$$+32\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}[\beta\delta(\alpha^2+\gamma^2)R'-\alpha\gamma(\beta^2+\delta^2)S']$$

$$+128\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}[\beta^2(\alpha^2+\beta^2+\gamma^2+\delta^2)Q'-2\alpha^2\beta\delta R'+2\beta^2\alpha\gamma S'],$$

where

$$P' = \frac{\sinh\frac{(2n+1)\pi}{2\alpha}(\eta-\delta)}{\cosh\frac{(2n+1)\pi\beta}{2\alpha}} \sin\frac{(2n+1)\pi}{2\alpha}(\xi-\gamma),$$

$$Q' = \frac{\sinh\frac{(2n+1)\pi}{2\beta}(\xi-\gamma)}{\cosh\frac{(2n+1)\pi\alpha}{2\beta}} \sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta),$$

$$R' = \frac{\cosh\frac{(2n+1)\pi}{2\alpha}(\eta-\delta)}{\sinh\frac{(2n+1)\pi\beta}{2\alpha}} \sin\frac{(2n+1)\pi}{2\alpha}(\xi-\gamma),$$

and
$$S' = \frac{\cosh\frac{(2n+1)\pi}{2\beta}(\xi-\gamma)}{\sinh\frac{(2n+1)\pi\alpha}{2\beta}} \sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta).$$

It may be observed that the results obtained for the case of three confocal parabolas can be obtained from those obtained in this article by putting $\xi_1 = -\xi_2$ (so that $\gamma = 0$) in this article and $\xi_1 = \alpha$ in §§ 2 and 3.

Another expression for ψ_1, with a corresponding expression for ϕ_1, is obtained by the assumption

$$\psi_1 = \sum_1^\infty \sin\frac{n\pi}{2\alpha}(\xi-\xi_1)\left[A_n\sinh\frac{n\pi}{2\alpha}(\eta-\eta_1) + B_n\sinh\frac{n\pi}{2\alpha}(\eta_2-\eta)\right]$$

$$+\sum_1^\infty \sin\frac{n\pi}{2\beta}(\eta-\eta_1)\left[C_n\sinh\frac{n\pi}{2\beta}(\xi-\xi_1) + D_n\sinh\frac{n\pi}{2\beta}(\xi_2-\xi)\right].$$

In this case we shall find

$$A_n = \frac{8\omega\eta_2^2\left[(-1)^n\left(\frac{8\alpha^2}{n^2\pi^2}-\xi_2^2\right)-\left(\frac{8\alpha^2}{n^2\pi^2}-\xi_1^2\right)\right]}{n\pi\sinh\frac{n\pi\beta}{\alpha}},$$

$$C_n = \frac{8\omega\xi_2^2\left[(-1)^n\left(\frac{8\beta^2}{n^2\pi^2}-\eta_2^2\right)-\left(\frac{8\beta^2}{n^2\pi^2}-\eta_1^2\right)\right]}{n\pi\sinh\frac{n\pi\alpha}{\beta}},$$

B_n and D_n are what A_n and C_n become when η_1 is written for η_2 in A_n and ξ_1 for ξ_2 in C_n.

Also

$$\phi_1 = -\sum_1^\infty \cos\frac{n\pi}{2a}(\xi-\xi_1)\left[A_n\cosh\frac{n\pi}{2a}(\eta-\eta_1)+B_n\cosh\frac{n\pi}{2a}(\eta_2-\eta)\right]$$

$$+\sum_1^\infty \cos\frac{n\pi}{2\beta}(\eta-\eta_1)\left[C_n\cosh\frac{n\pi}{2\beta}(\xi-\xi_1)+D_n\cosh\frac{n\pi}{2\beta}(\xi_2-\xi)\right].$$

A third and simpler expression for ψ_1 can be obtained by assuming

$$\psi_1 = \sum_1^\infty \sin\frac{n\pi}{a}(\xi-\gamma)\left[A_n\sinh\frac{n\pi}{a}(\eta-\eta_1)+B_n\sinh\frac{n\pi}{a}(\eta_2-\eta)\right]$$

$$+\sum_1^\infty \sin\frac{n\pi}{\beta}(\eta-\delta)\left[C_n\sinh\frac{n\pi}{\beta}(\xi-\xi_1)+D_n\sinh\frac{n\pi}{\beta}(\xi_2-\xi)\right].$$

It will be found that

$$A_n = \frac{(-1)^{n+1}}{n\pi}\cdot\frac{16\omega\eta_2^2 a\gamma}{\sinh\frac{2n\pi\beta}{a}},\quad B_n = \frac{(-1)^{n+1}}{n\pi}\cdot\frac{16\omega\eta_1^2 a\gamma}{\sinh\frac{2n\pi\beta}{a}},$$

$$C_n = \frac{(-1)^{n+1}}{n\pi}\cdot\frac{16\omega\xi_2^2\beta\delta}{\sinh\frac{2n\pi a}{\beta}},\quad D_n = \frac{(-1)^{n+1}}{n\pi}\cdot\frac{16\omega\xi_1^2\beta\delta}{\sinh\frac{2n\pi a}{\beta}}.$$

The corresponding expression for ϕ_1 can be easily deduced.

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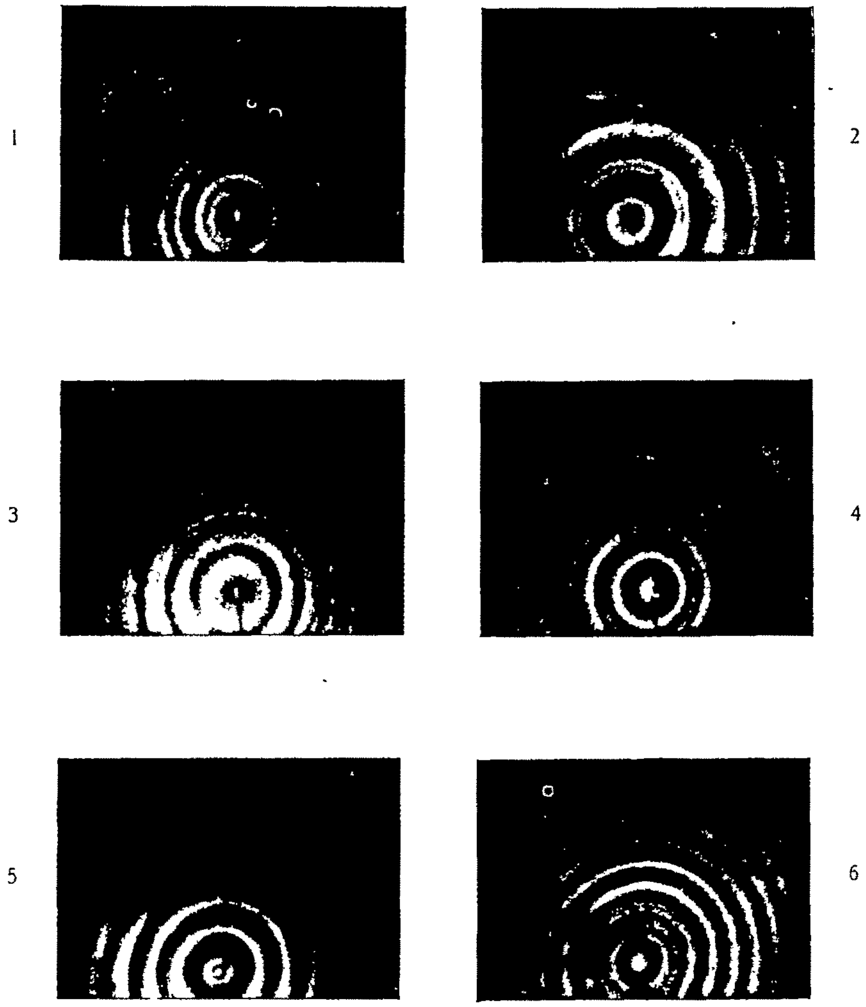

Ripples of Large Amplitude.

Investigations on Ripple Motion

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[Plate]

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The problem of wave motion on the surface of liquids has for a long time been the subject of both theoretical and experimental study. Amongst the earlier workers were Scott Russel,[1] Cauchy[2] and Poisson[3] ; the last two solved the problem of the effect of a local impulse on the surface of water. Soon after Stokes[4] gave a complete theory of surface waves of finite amplitude, without however taking the effect of surface tension into consideration, but Kelvin[5] gave the theory of capillary waves of small amplitude and proved the existance of a minimum velocity of propagation, which was afterwards experimentally verified by Rayleigh.[6] More recently Dr. Wilton[7] gave a theory of the form of ripples of finite amplitude and showed mathematically that the lengths of crests is greater than that of the hollows. While trying to verify Wilton's theory some remarkable results were obtained by the present author in the case of the larger wave lengths. It was found that for finite amplitudes the crests of waves begin to divide with increase of wave length, the form of the waves thus becoming complex with two or more crests in each wave length. It was also found that the form of the waves changes with the depth of liquid ; for small depths the form becomes simpler, but the depth at which this change of form takes place is very small. As might be expected the depth at which the change of form occurs was found to be less for small lengths and greater for larger wave lengths.

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[1] *Brit. Ass. Rep.*, Vol. VI, 1844.

[2] "Mémoire sur le théorie des ondes," *Mem. de l' Acad. Roy. des. Sciences*, t. 1. (1827).

[3] "Mémoire sur la théorie des oddes," *Mem.. de l' Acad. Roy. des Sciences*, t. 1. (1816), Lamb's Hydradynamics, 4th edition, p. 373).

[4] *Camb. Trans.*, t. VIII (1847) and *papers*, Vol. I, p. 197.

[5] *Baltimore Lectures*, p. 598 ; *Phil. Mag.* (4), t. XLVII, p. 374.

[6] *Phil. Mag.* (5), t. XXX, p. 386, *papers*, t. III, p. 394.

[7] *Phil. Mag.*, May, 1915, p. 688.

Ripples can be excited on the surface of a liquid either by periodic puffs of air directed against a point or by the periodic vertical motion of a dipper at a point on the surface of the liquid. The dipper can be attached to a tuning fork, but it was found more advantageous to attach it to Prof. Fleming's motor vibrator as improved by Prof. C. V. Raman.[1] By means of this instrument, we can vary the amplitude of the waves very conveniently and accurately, and also the wave length, by increasing or decreasing the speed of the motor by including a rheostat in its circuit. The ripples thus formed are seen by the method of stroboscopic observation. The waves thus seen, appear to be a series of concentric circles at equal distances in the case of small amplitudes and for large amplitudes and increasing wave lengths, the circles become double, triple and so on, corresponding to two or more crests for each wave length. Some photographs of those forms are shown in Figs. 1—6.

The phenomena observed above can be explained mathematically as follows. The simpler case of plane waves is taken up first.

Let the axis of x be in the direction of propagation of waves and the axis of y vertically upwards and the undisturbed level of the liquid as the plane $y=0$, and let the depth of the liquid be h. Then for irrotational two dimensional wave motion, the velocity potential satisfies the equation.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0, \qquad \ldots \quad (1)$$

$$\frac{\partial\phi}{\partial y}=0, \quad \text{when } y=-h. \qquad \ldots \quad (2)$$

This condition denotes that the velocity at the bottom is zero and the pressure condition at the surface gives,

$$-\frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}=-gy+\frac{\partial\phi}{\partial t}-\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}, \qquad \ldots \quad (3)$$

taking the curvature to be small, which is true for pretty long waves. Since the disturbing force is simple harmonic, we can suppose the following expression for ϕ,

$$\phi=\sum_{s=1}^{s=n} a_s \cosh sm\,(y+h)\sin s\,m\,(ct-x), \qquad \ldots \quad (4)$$

which satisfies (1) and (2)

We have also the condition,

$$-\frac{\partial\phi}{\partial y}=\frac{\partial y}{\partial t}\text{, and for } y=0,$$

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[1] *Phys. Rev.*, Nov., 1919.

this gives

$$y=\frac{1}{c}\sum_{s=1}^{s=n} a_s \sinh smh \cos s\,m\,(ct-x) \qquad \ldots \ (5)$$

To determine the constants a_1, a_2, a_3 etc, we substitute the values of ϕ and y in the pressure equation (3) and equate the co-efficients of $\cos sm(ct-x)$ to zero, we get approximate equations for the determination of c, a_2, a_3 .. etc, a_1 itself being arbitrary. For practical purposes it is sufficient to take only three terms. Thus we get the following values for c, a_2 and a_3 :

$$c^2=\left(\frac{Tm}{\rho}+g\right)\tanh mh-\frac{m^3}{2c}\,a_1{}^2\sinh^2 mh$$

$$-\tfrac{3}{2}\,m^3a_2\tanh mh\sinh^2 mh+\frac{m^2}{2}\,a_2\,\frac{\cosh 3\,mh}{\cosh mh}, \qquad \ldots \ (6)$$

$$a_2=\left(\frac{m^2}{2}\,a_1{}^2\sinh^2 mh-\frac{m^2}{4}\right)$$

$$\Big/\left\{\left(\frac{4Tm^2}{\rho c}+\frac{g}{c}\right)\sinh 2\,mh-2mc\cosh 2\,mh\right.$$

$$-\frac{2m^3}{c}\,a_1{}^3\sinh^2 mh\cosh 2\,mh-\frac{m^3}{2c}\,a_1{}^2\sinh mh\sinh 2mh\cosh mh$$

$$\left.+\tfrac{3}{2}\,\frac{m^3}{c}\,a_1{}^2\sinh mh\sinh 3mh\right\},$$

and $$a_3=\left(\tfrac{3}{2}\,m^2a_1a_2\sinh mh\sinh 2mh-\tfrac{3}{2}\,\frac{m^3}{c}\,a_1a_2{}^2\sinh mh\sinh 3mh\right.$$

$$\left.+\frac{m^3}{c}\,a_1\,a_2{}^2\sinh^2 2mh\right)$$

$$\Big/\left\{\left(\frac{9Tm^2}{\rho c}+\frac{g}{c}\right)\sinh 3mh-3mc^3\cosh 3mh\right.$$

$$-c^2\,m^3\,a_1{}^3\left(\tfrac{1}{2}\sinh 3mh\sinh^2 mh+\frac{27}{4}\cosh 3mh\sinh^2 mh\right.$$

$$\left.\left.-\tfrac{3}{2}\sinh 4mh\sinh mh\right)\right\},$$

Putting $\frac{a_1}{c}\sinh mh=a$, the formula for the wave form becomes

$$y=a\cos m\,(ct-x)$$

$$+\frac{\left(\dfrac{c^2m^2}{2}-\dfrac{m^2c^2}{4\sinh^3 mh}\right)\sinh 2mh\cos 2m\,(ct-x)}{\left(\dfrac{4Tm^2}{\rho}+y\right)\sinh 2mh-2mc^2\cosh 2mh-m^3a^2\left(2\cosh 2mh+\cosh^2 mh-\tfrac{3}{2}\dfrac{\cosh 3mh}{\cosh mh}\right)}.$$

$$+\left\{\tfrac{3}{2}cm^2aa_2\sinh 2mh+aa_2{}^2m^3\left(\frac{\sinh^2 2mh}{\sinh mh}-\tfrac{3}{2}\sinh 3mh\right)\right\}$$

$$\times\sinh 3mh\cos 3m\,(ct-x)$$

$$\bigg/\left\{\left(\frac{9Tm^2}{\rho}+g\right)\sinh 3mh-3mc^2\cosh 3mh-c^2m^2a^2\left(\tfrac{1}{2}\sinh 3mh\right.\right.$$

$$\left.\left.+\frac{27}{4}\cos 3mh-\tfrac{3}{2}\frac{\sinh 4mh}{\sinh mh}\right)\right\}, \qquad \ldots \quad (7)$$

where a_2 is the co-efficient in the second term of $\cos 2m\,(ct-x)$. This formula becomes for infinite depth,

$$y=a\cos m\,(ct-x)+\frac{\frac{m^2}{2}a^2c^2\cos 2m\,(ct-x)}{\left(\frac{4Tm^2}{\rho}+g\right)-2mc^2+\frac{1}{2}m^3a^2c^2-m^3a'^2c^2}$$

$$+\frac{\frac{3}{2}c^2m^2aa'-\frac{1}{2}c^2m^3aa'^2}{\frac{9Tm^2}{\rho}+g-3mc^2+\frac{3}{4}m^3c^2a^2+\frac{49}{4}m^3a'^2c^2}, \qquad \ldots \quad (8)$$

where
$$a'=\frac{\frac{m^2}{2}a^2}{\frac{4Tm^2}{\rho}+g-2mc^2+\frac{1}{2}m^3a^2c^2}.$$

This formula fully explains the phenomenon observed. For large gravity waves the formula reduces to

$$y=a\cos m(ct-x)-\tfrac{1}{2}ma^2\cos 2m\,(ct-x)+\tfrac{3}{8}m^2a^3\cos 3m\,(ct-x).$$

This shows that the form of the waves becomes cycloidal as was pointed out by Stokes.

To explain the change of form of waves with increasing distance from the source, we use cylindrical coordinates. We take the axis of z vertically upwards and as before the undisturbed surface of the water as the plane $z=0$.

The equation are modified as

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0, \qquad \ldots \quad (9)$$

$$\frac{\partial\phi}{\partial r}=0,\ \text{when } r=\infty,$$

$$\frac{\partial\phi}{\partial z}=0,\ \text{when } z=-h. \qquad \ldots \quad (10)$$

The pressure equation becomes

$$-\frac{T}{\rho}\frac{\partial^2\phi}{\partial r^2}=-gz+\frac{\partial\phi}{\partial t}-\tfrac{1}{2}\left\{\left(\frac{\partial\phi}{\partial z}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2\right\}. \quad \ldots \quad (11)$$

We can therefore assume

$$\phi=\sum_{s=1}^{s=n} a_s\, J_0\,(skr)\cosh sk\,(z+h)\sin skct, \quad \ldots \quad (12)$$

and from the condition $-\frac{\partial\phi}{\partial z}=\frac{\partial z}{\partial t}$, we get for $z=0$,

$$z=\frac{1}{c}\sum_{s=1}^{s=n} a_s\, J_0\,(skr)\sinh skh\cos skct. \quad \ldots \quad (13)$$

Substituting these values in the pressure equation (11) and taking only three terms into consideration as in the case of plane waves, and equating the co-efficients of $\cos sm(ct-x)$ to zero, we get equations for the determination of c, a_2 and a_3, a_1 itself being arbitrary.

Thus we get

$$c^2=-\frac{Tk}{\rho}\frac{J_0''(kr)}{J_0(kr)}\tanh kh+\frac{g}{k}\tanh kh-\frac{k}{2}a_1^2\{J_0(kr)\}^2$$

$$+k^2a_2\frac{J_1(kr)}{J_0(kr)}\,.\,J_1\,(2kr)\cosh 2kh-\tfrac{3}{2}\,\frac{k^2}{c}\,a_2\,J_0(2kr)\tanh kh\sinh 2kh,$$

since

$$J_0''(kr)=-J_0(kr)+\frac{1}{kr}J_1(kr),$$

we have as a first approximation

$$c^2=\left(-\frac{Tk}{\rho}\frac{J_0''(kr)}{J_0(kr)}+\frac{g}{k}\right)\tanh kh$$

$$=\left(\frac{Tk}{\rho}-\frac{Tk}{\rho}\frac{1}{kr}\frac{J_1(kr)}{J_0(kr)}+\frac{g}{k}\right)\tanh kh$$

and

$$z=\frac{a_1}{c}\sinh kh\,J_0'\,(kr)\cos kct$$

$$+\frac{1}{c}\left(\left[\tfrac{3}{4}k^2a_1^2\{J_0(kr)\}^2\sinh^2 kh+\frac{k^2a_1^2}{4}\{J_1(kr)\}^2\cosh^2 kh\right]\right.$$

$$\left.\sinh 2kh\,J_0\,(2kr)\cos 2kct\right)\Bigg/\left(-\frac{4Tk^2}{\rho}J_0''(2kr)\sinh 2kh\right.$$

$$\left.+\frac{g}{c}J_0\,(2kr)\sinh 2kh-2kc\,J_0\,(2kr)\cosh 2kh\right)$$

$$+\left\{\frac{1}{c}\left(\tfrac{3}{2}a_1a_2J_0(2kr)\sinh kh\sinh 2kh-a_1a_2J_1(2kh)\cosh kh\cosh 2kh\right\}\right.$$

$$\times\sinh 3kh\,J_0\,(3kr)\cos 3kct$$

$$\Big/ \left(\frac{9Tk^2}{\rho c}\ J_o''(3kr) \sinh 3kh + \frac{g}{c}\ J_o(3kr) \sinh 3kh \right.$$
$$-3kc\ J_o(3kr) \cosh 3kh - \tfrac{3}{2}\, a_1 a_2, \qquad \ldots \quad (14)$$

when a_2 is the co-efficient of the 2nd term.

Putting $\frac{a_1}{c}$ sinh $kh = a$ and simplifying we get,

$$z = a\ J_o(kr) \cos kct$$
$$+ \frac{\left[\tfrac{3}{4}c^2k^2a^2\{J_o(kr)\}^2 + \frac{k^2c^2a^2}{4}\ J_1(kr)\}^2\ \cosh^2 kh\right] \sinh 2kh \cos 2kct}{-\frac{4Tk^2}{\rho}\ \frac{J_o''(2kr)}{J_o(2kr)} \sinh 2kh + g \sinh 2kh - 2kc^2 \cosh 2kh}$$

$+$etc.

For infinite depth, this becomes

$$z = a\ J_o(kr) \cos kct$$
$$+ \frac{\tfrac{3}{4}\, c^2k^2a^2\{J_o(kr)\}^2 + \frac{c^2k^2a^2}{4}\ \{J_1(kr)\}^2}{\left(-\frac{4Tk^2}{\rho}\ \frac{J_o''(2kr)}{J_o(2kr)} + g\right) - 2kc^2 + \ldots} \cos 2kct$$

$+$etc.

The asymptotic value of $J_o(kr)$ is

$$\sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{\pi}{4}\right),$$

and that of $J_1(kr)$ is $\sqrt{\frac{2}{\pi kr}} \sin\left(kr - \frac{\pi}{4}\right)$.

Substituting these values in the above formula, we get

$$z = a\sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{\pi}{4}\right) \cos kct$$
$$+ \frac{2}{\pi kr}\ \frac{\tfrac{1}{4}k^2a^2(1 + 2 \sin 2kr) \cos 2kct}{-\frac{4Tk^2}{\rho}\ \frac{J_o''(2kr)}{J_o(2kr)} + g - 2kc^2} + \text{etc.}$$

The above formula shows that the amplitude of the second harmonic decreases inversely as the distance from the origin, whereas the fundamental decreases inversely as the square root of the distance. Hence the wave form would tend to become simpler with the increase of distance from the source and this is what is also found experimentally.

Note on the product of any number of Legendre functions of different degrees

By

Sudhansukumar Banerji

[*Read September 4th, 1920*]

In 1878, Adams published in the Proceedings of the Royal Society of London, the well known result on the product of two Legendre functions of different degrees, namely

$$P_m(z)\, P_n(z) = \sum_{r=0}^{m} \frac{2n+2m-4r+1}{2n+2m-2r+1} \cdot \frac{A(n-r)A(m-r)A(r)}{A(n+m-r)}\, P_{n+m-2r}(z)$$

where

$$A(m) = \frac{1.3.5\ldots(2m-1)}{m\,!}, \qquad \ldots \quad (1)$$

m and n being positive integers and $m \leq n$.

This result can also be written in the form

$$P_m(z)\, P_n(z) = \sum_{q=n-m}^{n+m} \frac{2q+1}{m+n+q+1} \cdot \frac{A\left(\frac{n+m-q}{2}\right) A\left(\frac{n+q-m}{2}\right) A\left(\frac{m+q-n}{2}\right)}{A\left(\frac{n+m+q}{2}\right)}\, P_q(z)$$

$$= \sum_{q=n-m}^{n+m} A_{n,m,q}\, P_q(z), \qquad \ldots \quad (2)$$

where

$$A_{n,m,q} = \frac{2q+1}{m+n+q+1} \cdot \frac{A\left(\frac{n+m-q}{2}\right) A\left(\frac{n+q-m}{2}\right) A\left(\frac{m+q-n}{2}\right)}{A\left(\frac{n+m+q}{2}\right)}$$

and the summation extends for the following values of q

$$n-m,\ n-m+2,\ \ldots n+m.$$

From this Adams deduced the value of the integral

$$\int_{-1}^{1} P_m(z)\, P_n(z)\, P_q(z)\, dz$$

$$= \frac{2}{m+n+q+1} \frac{A\left(\frac{n+m-q}{2}\right) A\left(\frac{m+q-n}{2}\right) A\left(\frac{q+n-m}{2}\right)}{A\left(\frac{n+m+q}{3}\right)} \qquad \ldots \quad (3)$$

and q takes the values

$$n-m, \quad n-m+2, \quad n-m+4, \ldots n+m, \quad \text{if } n>m$$

or

$$m-n, \quad m-n+2, \quad m-n+4, \ldots m+n, \quad \text{if } m>n$$

and for other values of q the integral is zero.

The method can easily be extended to obtain an expression in a series of Legendre functions for the product of three or more Legendre functions and when this has been done the values of similar integrals involving four or more Legendre functions can be at once written down. It is interesting to note the results in the different cases and the conditions under which they hold.

If (2) be multiplied by $P_s(z)$, where s is a positive integer and if $s \geq n+m$, we have

$$P_m(z)\, P_n(z)\, P_s(z)$$

$$= \sum_{q=n-m}^{n+m} P_q(z)\, P_s(z)$$

$$= \sum_{q=n-m}^{n+m} A_{n,m,q} \sum_{t=s-q}^{s+q} A_{s,q,t}\, P_t(z)\,, \qquad \ldots \quad (4)$$

$$= A_{n,m,n+m}\, A_{s,n+m,s-n-m}\, P_{s-n-m}(z)$$
$$+ (A_{n,m,n+m-2}\, A_{s,n+m-2,s+2-n-m} + A_{n,m,n+m}\, A_{s,n+m,s+2-n-m})\, P_{s+2-n-m}(z)$$
$$+ (A_{n,m,n+m-4}\, A_{s,n+m-4,s+4-n-m} + A_{n,m,n+m-2}\, A_{s,n+m-2,s+4-n-m} + A_{n,m,n+m}\, A_{s,n+m,s+4-n-m})\, P_{s+4-n-m}(z)$$
$$+ \ldots + (A_{n,m,n+m}\, A_{s,n+m,s+n+m})\, P_{s+n+m}(z)\,. \qquad \ldots \quad (5)$$

From this the law of formation of the co-efficients is obvious.

If $s \leq n-m$, we have

$$P_m(z)\,P_n(z)\,P_s(z) = \sum_{q=n-m}^{n+m} A_{n,m,q} \sum_{t=q-s}^{q+s} A_{q,s,t}\,P_t(z)$$

$$= A_{n,n,n-m}\,A_{n-m,s,n-m-s}\,P_{n-m-s}(z)$$

$$+ (A_{n,m,n-m+2}\,A_{n-m+2,s,n-m-s+2} + A_{n,m,n-m}\,A_{n-m,s,n-m-s+2})\,P_{n-m-s+2}(z)$$

$$+ (A_{n,m,n-m+4}\,A_{n-m+1,s,n-m-s+4} + A_{n,m,n-m+2}\,A_{n-m+2,s,n-m-s+4}$$
$$+ A_{n,m,n-m}\,A_{n-m,s,n-m-s+4})\,P_{n-m-s+4}(z)$$

$$+ \ldots + A_{n,m,n+m}\,A_{n+m,s,n+m+s}\,P_{n+m+s}(z).$$

If $n-m<s<n+m$, let $s=n-m+r$, where r may be an odd or an even integer $(0<r<2m)$.

If r is an odd integer, then

$$P_n(z)\,P_m(z)\,P_s(z)$$
$$= \sum_{q=n-m}^{n-m+r-1} A_{n,m,q} \sum_{t=n-m+r-q}^{n-m+r+q} A_{n-m+r,q,t}\,P_t(z)$$
$$+ \sum_{q=n-m+r+1}^{n+m} A_{n,m,q} \sum_{t=q-n+m-r}^{q+n-m+r} A_{q,n-m+r,t}\,P_t(z).$$

If r is an even integer, then

$$P_n(z)\,P_m(z)\,P_s(z)$$
$$= \sum_{q=n-m}^{n-m+r} A_{n,m,q} \sum_{t=n-m+r-q}^{n-m+r+q} A_{n-m+r,q,t}\,P_t(z)$$
$$+ \sum_{q=n-m+r+2}^{n+m} A_{n,m,q} \sum_{t=q-n+m-r}^{q+n-m+r} A_{q,n-m+r,t}\,P_t(z).$$

These cover all the cases that may arise for the product of three Legendre functions.

To evaluate the integral

$$\int_{-1}^{1} P_n(z)\,P_m(z)\,P_s(z)\,P_t(z)\,dz,$$

it should be noticed that if $s>n+m$, then the integral will have a definite value other than zero if t takes any of the values

$$s-n-m, \ldots s-n-m+2, \quad s-n-m+4, \ldots s+n+m,$$

and for other values of t, the integral is zero.

When t has any of the values given above the value of the integral can be written in the form

$$\frac{2}{2t+1} \sum_{r} A_{n,m,n-m+2r} \sum_{r'} A_{s,n-m+2r,\ s-n+m+2(r-r')}$$

where the first summation extends for all integral values of r form 0 to m and the second summation for all integral values of r' from 0 to $n-m+2r$, which make

$$r-r'=\text{a given integer}=\frac{(s+m)-(n+t)}{2}.$$

The following particular cases are interesting :

$$\int_{-1}^{1} P_n(z)\, P_m(z)\, P_s(z)\, P_{s-n-m}(z)dz$$

$$=\frac{2}{2s-2n-2m+1}\ A_{n,m,n+m}\, A_{s,n+m,s-n-m}$$

$$\int_{-1}^{1} P_n(z)\, P_m(z)\, P_s(z)\, P_{s-n-m+2}(z)dz$$

$$=\frac{2}{2s-2n-2m+5}\ \{A_{n,m,n+m-2}\, A_{s,n+m-2,s+2-n-m} + A_{n,m,n+m}\, A_{s,n+m,s+2-n-m}],$$

$$\int_{-1}^{1} P_n(z) P_m(z) P_s(z) P_{s-n-m+4}(z)\, dz$$

$$=\frac{2}{2s-2n-2m+9}\ [A_{n,m,n+m-4}\, A_{s,n+m-4,s+4-n-m} + A_{n,m,n+m-2}\, A_{s,n+m-2,s+4-n-m} + A_{n,m,n+m}\, A_{s,n+m,s+4-n-m}],$$

etc.

When $s<n-m$, the integral

$$\int_{-1}^{1} P_n(z)\, P_m(z)\, P_s(z)\, P_t(z)dz$$

has a value other than zero when t takes any of the values

$$n-m-s,\quad n-m-s+2,\quad n-m-s+4,\ldots n+m+s,$$

and for other values of t, the integral is zero.

The value of the integral when t has any of the above values can be easily written down. The following particular cases are interesting :

$$\int_{-1}^{1} \mathrm{P}_n(z)\, \mathrm{P}_m(z)\, \mathrm{P}_s(z)\, \mathrm{P}_{n-m-s} dz$$

$$= \frac{2}{2n-2m-2s+1} \cdot \mathrm{A}_{n,m,n-m}\, \mathrm{A}_{n-m,s,n-m-s}.$$

$$\int_{-1}^{1} \mathrm{P}_n(z)\, \mathrm{P}_m(z)\, \mathrm{P}_s(z)\, \mathrm{P}_{n-m-s+2}(z) dz$$

$$= \frac{2}{2n-2m-2s+5} \left[\mathrm{A}_{n,m,n-m+2}\, \mathrm{A}_{n-m+2,s,n-m-s+2} + \mathrm{A}_{n,m,n-m}\, \mathrm{A}_{n-m,s,n-m-s+2}\right],$$

etc.

If $n-m<s<n+m$, let $s=n-m+r$, where r may be an odd or even integer $(0<r<2m)$. If r is an odd integer, then the integral

$$\int_{-1}^{1} \mathrm{P}_n(z)\, \mathrm{P}_m(z)\, \mathrm{P}_s(z)\, \mathrm{P}_t(z) dz$$

will have a value other than zero, if t takes any of the values

$$1,\ 3,\ 5, \ldots 2n-2m+2r-1, \ldots 2n+r,$$

and for other values of t, the integral is zero.

If however r is an even integer, then the integral has a value other than zero if t takes any of the values

$$0\ \ 2,\ 4, \ldots 2n-2m+2r, \ldots 2n+r,$$

and for other values of t the integral is zero.

The value of the integral when t has any of the values enumerated above can be easily written down.

The method can obviously be extended to the product of four and more Legendre functions of different degrees but the conditions under which the diffierent expressions hold become more and more numerous and the results more and more complicated.

As an application of the integral of the product of three Legendre functions of different degrees, it may be mentioned here that it was shown by the author in a previous paper published in this Bulletin that if (r, θ) and (r', θ') represent the polar co-ordinates of a point measured from two given poles, then

$$\frac{\mathrm{J}_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\ \mathrm{P}_n(\mu)=\sum_{p=0}^{\infty} \mathrm{C}_{n,p}\ \frac{\mathrm{J}_{p+\frac{1}{2}}(kr')}{\sqrt{r'}}\ \mathrm{P}_p(\mu'),$$

where $\cos\theta=\mu$, $\cos\theta'=\mu'$ and

$$\mathrm{C}_{n,p}=(2p+1)i^{p+n}\int_{-1}^{1} \mathrm{P}_p(z)\,\mathrm{P}_n(z)\,e^{-ik\,\mathrm{D}}\,dz$$

$$=(2p+1)i^{p+n}\sqrt{\frac{\pi}{2k\mathrm{D}}}\sum_{s=0}^{\infty} i^{3s}\,(2s+1)\,\mathrm{J}_{s+\frac{1}{2}}(k\mathrm{D})\int_{-1}^{1} \mathrm{P}_p(z)\,\mathrm{P}_n(z)\,\mathrm{P}_s(z)dz$$

$$=(2p+1)i^{p+n}\sqrt{\frac{\pi}{2k\mathrm{D}}}\sum_{s} i^{3s}(2s+1)\,\mathrm{J}_{s+\frac{1}{2}}(k\mathrm{D})\,\mathrm{A}_{p,n,s}\ \frac{2}{2s+1},$$

D being the distance between the poles and the summation with respect to s extends from $p-n$ to $p+n$ or $n-p$ to $p+n$.

For this and similar results reference is made to two papers published by the author in the Bulletin of the Society, Vol. IV and Vol. V.

Other applications of the results obtained in this paper will be published in due course.

Notes

The International Congress of Mathematicians (1920) was held at Strasbourg, September, 22-30, 1920. The first and fifth days, and the last two days, were reserved for excursions. There were four general lectures by Volterra, Larmor, Nörlund and Dickson. The papers were read in each of four sections. In Section I (Arithmetic, Algebra, Analysis) there were 33 papers; in Section II (Geometry), 13 papers; in Section III (Mechanics, mathematical physics, applied mathematics), 23 papers; and in Section IV (Philosophical, historical and pedagogical questions), 10 papers. Only three Englishmen took part in the programme: Larmor, Young and Greenhill. Italians, Spaniards and Scandinavians, as well as Germans and Austrians were conspicuously absent. The total attendance was above 200. The International Mathematical Union voted to hold the International Mathematical Congress of 1924 in the United States of America.

On page 59, Vol. XI of the Bulletin of the Calcutta Mathematical Society, in dealing with Sylvester's Law of Nullity, the Reviewer of Dr. Cullis's Matrices and Determinoids, Vol. I stated that Dr. Cullis's Theorem III of Art. 71 gave only *one half* of Sylvester's famous law. The reviewer is glad to see that in Chapter XV of the Second Volume, Dr. Cullis has dealt very *completely* with the ranks of matrix products and, in particular, in Art. 33 of that chapter, has given the most general theorem of which Sylvester's Law is only a *particular* case.

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Sur divers modes de convergence d'une suite de fonctions d'une variable[1]

PAR

MAURICE FRÉCHET [STRASBOURG]

[*Read, January 16th, 1921*]

1. *Introduction.* J'ai indiqué dans ma Thèse, et depuis lors, comment on pouvait faire correspondre à certaines définitions classiques de la limite d'une suite d'éléments, des définitions convenables de la "distance" de deux éléments, satisfaisant aux conditions suivantes :

(*i*) a tout couple a, b d'éléments est attaché un nombre $(a, b)=(b, a) \geq 0$, nombre qui est appelé distance des deux éléments

(*ii*) deux éléments sont considérés comme identiques si $(a, b)=0$

(*iii*) si a, b, c sont trois éléments quelconques, on a $(a, b) \leq (a, c) + (c, b)$

(*iv*) la condition nécessaire et suffisante pour qu'une suite d'éléments $a_1, a_2, \ldots a_n, \ldots$ converge vers a est que (a_n, a) converge vers zéro.

Dans ma Thèse, j'employais au lieu du mot "distance," l'expression "écart" que j'ai réservée, depuis, au cas plus général où la condition *iii* n'est pas imposée.

Je me propose de considérer ici plus particulièrement le cas où les éléments considérés sont des fonctions d'une variable réelle définies dans un même intervalle I.

Première Partie.

2. *Convergence relativement uniforme.* Le Professeur E. H. Moore a introduit une notion qui généralise la notion de convergence uniforme. D'après lui,[2] une suite de fonctions $f_n(x)$ converge

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[1] Communicated by Prof. S. K. Banerji.

[2] Voir par exemple : *Introduction to a Form of General Analysis*, 1910, page 30.

uniformément vers $f(x)$ sur l'intervalle I relativement à une fonction $\sigma(x)$ lorsque à tout nombre $\epsilon > 0$ on peut faire correspondre un entier p tel que l'on ait sur I

$$|f_n(x)-f(x)| \leq \epsilon\,|\sigma(x)| \qquad \text{pour } n > p.$$

Lorsque $|\sigma(x)|$ a une forme supérieure fixée et une borne inférieure positive la convergence est uniforme au sens ordinaire. La notion de convergence relativement uniforme a été formulée pour introduire le cas nouveau où $\sigma(x)$ n'est pas bornée.

On remarque tout de suite qu'elle se ramène à la convergence uniforme si l'on opère sur le champ des fonctions $f(x)$ considérée la transformation

$$F(x) = \frac{f(x)}{\sigma(x)}$$

dans le cas le plus général où $|\sigma(x)|$ est borné ou non pourvu que $\sigma(x)$ soit partout $\neq 0$. De sorte que la convergence relativement uniforme est moins distincte qu'il ne parait de la convergence uniforme tout court.

3. Mais le point essentiel que je voulais signaler—bien qu'il soit à peu près évident,—c'est qu'*on peut définir une "distance,"* satisfaisant aux conditions *i, ii, iii, iv* mentionnées plus haut, lorsque les éléments considérés sont des fonctions d'une variable et lorsque la limite d'une suite d'éléments est assujettie à être *relativement uniforme,* par rapport à une fonction $\sigma(x)$ quelconque mais déterminée.

Il suffit pour cela d'appeler distance de deux fonctions $f(x)$, $\phi(x)$, (relativement à $\sigma(x)$) la borne inférieure (f, ϕ) des valeurs de ϵ telles que l'on ait dans tout l'intervalle I.

$$|f(x)-\phi(x)| \leq \epsilon\,|\sigma(x)|$$

Les conditions *i, ii, iv* sont évidemment vérifiées.[1] Quant à *iii,* on observe que

$$|f(x)-\phi(x)| \leq |f(x)-\psi(x)| + |\psi(x)-\phi(x)| \leq (f, \psi)\,|\sigma| + (\psi, f)\,|\sigma|\,;$$

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[1] Si l'on veut éviter que (f, ϕ) prenne des valeurs infinies, on peut le remplacer par $\dfrac{(f, \phi)}{1+(f, \phi)}$.

on a donc

$$(1) \qquad |f(x)-\phi(x)| \leq \epsilon\,|\sigma(x)|,$$

pour $$\epsilon \geq (f, \psi) + (\psi, \phi).$$

Par suite (f, ϕ) qui est la borne inférieure de toutes les valeurs de ϵ satisfaisant à (1) sera au plus égal à $(f, \phi) + (\psi, \psi)$.

4. La définition de E. H. Moore contient en germe l'idée de la convergence *par rapport à des fonctions majorantes.* Exprimons cette nouvelle notion dans toute sa généralité: attachons à chaque fonction $f(x)$ un ensemble de fonctions majorantes $Q^f_\epsilon(x)$; on dira que la suite $f_n(x)$ converge vers $f(x)$ par rapport aux fonctions $Q^f_\epsilon(x)$ si quelque soit le nombre $\epsilon > 0$ on a

$$|f_n(x)-f(x)| \leq |Q^f_\epsilon(x)|,$$

à partir d'un certain rang p indépendant de x.

Si de plus la borne inférieure de $|Q^f_\epsilon(x)|$ est zéro quand ϵ varie pour chaque valeur particulière de x, la convergence par rapport aux Q^f_ϵ impliquera la convergence ordinaire.

Le cas intéressant est d'ailleurs celui où la borne supérieure des valeurs de $|Q^f_\epsilon(x)|$ sur l'intervalle I est une fonction de ϵ dont la borne inférieure n'est pas nulle, sans quoi la convergence se réduirait à la convergence uniforme tout court.

5. Il est remarquable que la notion de distance ou d'écart de deux fonctions conduit naturellement à la conception de cette convergence relativement uniforme sous l'aspect généralisé qui vient d'être décrit.

6. *Influence de la notion "d'écart" sur la définition de la convergence.* Supposons qu'une définition de la convergence d'une suite de fonctions d'une variable ait été donnée de façon quelconque pourvu que d'une part elle implique au moins la convergence individuelle en chaque point x et que d'autre part elle puisse être formulée par l'intermédiaire d'une définition convenable de "l'écart" satisfaisant aux conditions, *i, ii, iv.* Nous allons mettre en évidence une propriété importante d'une telle convergence.

Nommons en effet $P_{\epsilon}^{f}(x)$ la borne supérieure pour x et n donnés, de $|f(x)-\phi(x)|$ où $\phi(x)$ est une quelconque des fonctions telles que la distance (f, ϕ) reste inférieure à $\epsilon>0$. Cette borne supérieure peut être infinie; mais pour une abscisse déterminée x_0, elle ne peut être infinie quelque soit le nombre positif ϵ; et même elle tend vers zéro avec ϵ. En effet si P_{ϵ} était infinie quelque soit ϵ et en particulier si elle ne tendait pas vers zéro avec ϵ, on pourrait former une suite de nombres ϵ_n tendant vers zéro et tel que $P_{\epsilon_n}(x_0)$ fut infini ou au moins supérieur à un nombre positif déterminé θ. Il y aurait donc une fonction $\phi_n(x)$ vérifiant les inégalités

$$(f, \phi_n) < \epsilon_n \qquad\qquad |f(x_0)-\phi_n(x_0)| > \theta > 0.$$

Mais, quand n croit indéfiniment, ϵ_n et par conséquent (f, ϕ_n) tendant vers zéro; donc ϕ_n converge vers f *en tout point* et en particulier pour $x=x_0$, contrairement à la dernière égalité.

D'ailleurs, d'après la définition même de $P_{\epsilon}^{f}(x)$, cette fonction ne peut croître quand ϵ décroit. En définitive nous voyons qu'à toute fonction $f(x)$ est attachée une fonction de x, $P_{\epsilon}^{f}(x)$ qui peut être infinie mais qui lorsque ϵ décroit vers zéro reste finie à partir d'une valeur suffisamment petite de ϵ, et même tend vers zéro sans jamais croître.

Inversement si (f_n, f) tend vers zéro, à tout nombre ϵ correspond un entier p, tel que sur tout l'intervalle I, on ait

$$|f_n(x)-f(x)| \leq P_{\epsilon}^{f}(x) \qquad\qquad \text{pour } n>p.$$

On voit donc que toute convergence de l'espèce envisagée *implique* une convergence uniforme relativement à la fonction $P_{\epsilon}^{f}(x)$. Mais cette convergence est plus générale que celle qui est définie sous ce nom par E. H. Moore et où $P_{\epsilon}^{f}(x)$ a la forme particulière $\epsilon\,|\sigma(x)|$.

Nous verrons d'ailleurs dans un moment que la convergence actuellement considérée si elle implique cette convergence relativement uniforme généralisée, peut ne pas s'y réduire. Mais, il y a lieu de déduire de l'existence des fonctions P_{ϵ}^{f} une conséquence importante que j'avais déduite dans ma Thèse des propriétés des classes de fonctions de Baire, mais qui peut s'obtenir ici d'une façon tout à fait élémentaire.

Il est impossible de formuler une définition de "l'écart" de deux fonctions $f(x)$, $\phi(x)$, satisfaisant aux conditions *i, ii, iii* mentionnées plus haut *et fournissant une définition de la convergence d'une suite de fonctions se réduisant à la convergence ordinaire.*

Pour le démontrer, considérons une suite infinie de point distinct de l'intervalle I : $x_1, x_2, \ldots\ldots x_n, \ldots\ldots$ et appelons $f_n(x)$, une fonction égale partout à $f(x)$ sauf au point x_n. Quelleque soit la valeur qu'elle prend en x_n, il est évident que $f_n(x)$ converge ordinairement vers $f(x)$, c'est à dire converge vers $f(x)$ vers chaque point x individuel même si celui-ci appartient à la suite $x_1, x_2, \ldots$.

Si l'on peut prendre pour $f_n(x_n)$ une valeur telle que

$$(2) \qquad |f_n(x_n)-f(x_n)| > P^f_{\epsilon_0}(x_n),$$

ϵ_0 était un nombre positif convenablement choisi, la fonction f_n sera nécessairement telle que

$$(f_n, f) > \epsilon_0$$

de sorte que si le nombre ϵ_0 est indépendant de n, la distance (f_n, f) ne peut converger vers zéro et la condition *iii* ne peut être satisfaite.

La seule circonstance qui puisse empêcher de réaliser l'inégalité (2), c'est celle où pour toute valeur positive de ϵ, l'une au moins des quantités

$$P^f_\epsilon(x_1), \quad P^f_\epsilon(x_2), \ldots \ldots$$

serait infinie et ceci quelleque soit la suite de nombres distincts

$$x_1, x_2, \ldots\ldots x_n, \ldots\ldots$$

Or pour x fixe et quelconque $P^f_{\frac{1}{p}}(x)$ est fini lorsque p est un entier assez grand. Si les points où $P^f_{\frac{1}{p}}(x)$ est fini restaient en nombre fini quelque soit p, leur ensemble, quand p varie, serait dénombrable et devrait pourtant constituer tout l'intervalle I qui comme on le sait n'est pas dénombrable. Il y a donc un entier p_0 tel que les points où $P^f_{\frac{1}{p_0}}(x)$ est fini soient en nombre infini. Il suffira dès lors de prendre parmi ceux-ci la suite $x_1, x_2, \ldots\ldots$ et de poser $\epsilon_0 = \frac{1}{p_0}$.

8. En modifiant très légèrement le raisonnement nous pouvons même aller plus loin et substituer dans l'énoncé à la convergence ordinaire la convergence "quasi uniforme." Cette notion a été introduite (sous le nom de convergence à traits) par Arzela qui l'a obtenu comme condition nécessaire et suffisante pour qu'une série de fonctions continues converge vers une fonction continue. Une fonction $\phi_n(x)$ converge quasi uniformément vers une fonction $\phi(x)$ dans un intervalle I si à tout entier N et à tout nombre $\epsilon > 0$ correspond un entier $N' \geq N$ tel que l'on ait

$$| \phi_n(x) - \phi(x) | < \epsilon$$

pour au moins une valeur de n comprise entre N et N' mais pouvant varier avec x.

Ceci étant rien ne nous empêche dans le raisonnement précédent de modifier légèrement les fonctions $f_n(x)$ de façon à rendre $f_n - f$ continue. Il suffira de représenter $f_n - f$ par une ligne brisée se confondant avec l'axe ou sauf dans un petit intervalle $x_n - \theta_n$, $x_n + \theta_n$ où elle formera un angle dont le sommet d'abscisse x_n aura une ordonnée $f_n(x_n) - f(x_n)$ déterminée comme plus haut. On pourra alors toujours choisir les x_n et θ_n pour que $f_n(x) - f(x)$ converge vers zéro en chaque point x. Comme $f_n - f$ est maintenant une fonction continue qui tend vers la fonction continue identiquement nulle, la convergence est bien quasi uniforme. Quant au choix des x_n, θ_n, on peut le fixer comme suit. On extraira de la suite infinie des x_n obtenue plus haut une suite n'ayant qu'un seul point limite. Puis on extraira de cette suite, une suite placée d'un seul côté de ce point limite et s'en rapprochant sans cesse. Alors il suffira de prendre pour θ_n par exemple le plus petit des nombres $\frac{| x_n - x_{n-1} |}{3}$, $\frac{| x_{n+1} - x_n |}{3}$. On voit qu'alors $f_n(x) - f(x)$ sera nul en dehors de l'intervalle limité par le point limite des x_n et par x_{n-1}; il convergera donc bien vers zéro partout.

Nous voyons donc qu'il *est impossible de formuler une définition de "l'écart" de deux fonctions* $f(x)$, $\phi(x)$ satisfaisant aux conditions *i, ii, iv* mentionnées plus haut et fournissant une définition de la convergence d'une suite de fonctions qui soit *équivalente à la convergence quasi-uniforme.*[1]

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[1] On voit ainsi que le théorème de ma Thèse (§ 22, page 15) non seulement se trouve démontré d'une façon élémentaire mais qu'il reçoit une double généralisation, la convergence ordinaire étant remplacée par la convergence quasi uniforme et le voisinage etant remplacé par l'écart (au sens de la note imprimée plus haut.)

L'énoncé reste encore exact si l'on n'exige de définir la distance que dans le champ des fonctions continues. Il faudra seulement dans ce cas que la fonction P_ϵ^f soit définie comme la borne de $| f(x)-\phi(x) |$ où f et ϕ sont deux fonctions *continues* telles que $(f,\phi) < \epsilon$.

9. *Exemple d'une définition de la "distance" de deux fonctions conduisant à une définition de la convergence qui implique la convergence ordinaire sans se réduire à la convergence relativement uniforme.* Comme nous avons vu que ni la convergence ordinaire, ni même la convergence quasi uniforme ne peuvent être déduites ni d'une définition de l'écart, ni par conséquent d'une définition de la distance de deux fonctions ; comme d'autre part nous avons vu qu'on pouvait déduire d'une définition convenable de la distance non seulement la convergence uniforme mais encore la convergence relativement uniforme, il est intéressant de montrer que de la notion de distance on peut déduire des modes de convergence ne se réduisant pas à la convergence relativement uniforme et impliquant pourtant la convergence ordinaire (sans s'y réduire).

Il nous suffira pour cela d'appeler distance de deux fonctions $f(x)$, $\phi(x)$ la quantité $(f, \phi)=[f, \phi] + ((f, \phi))$

avec
$$[f,\psi]=\frac{K_{(f-\phi)}}{1+K_{(f-\phi)}}\ ;\quad ((f,\psi))=\frac{A_{(f-\phi)}}{1+A_{(f-\phi)}}$$

en appelant K_ψ la borne supérieure finie ou non de $\left| \frac{\psi(x)}{k(x)} \right|$, $k(x)$ désignant une fonction fixe non bornée et partout >0, et en appelant A_ψ l'aire limitée par l'axe des x, les ordonnées $x=a$, $x=b$ qui limitent l'intervalle I et la courbe $y= | \psi(x) |$. Pour éviter toute ambiguité dans le cas où $\psi(x)$ ne serait pas une fonction continue, nous supposons l'aire A_ψ définie de la façon suivante. On divise l'intervalle (a, b) en un certain nombre fini d'intervalles partiels qui servent de base à des rectangles parallèles aux axes, limités à $0x$ et de hauteurs assez grandes (au besoin infinies) pour que l'ensemble de ces rectangles contienne toute la courbe $y=\psi(x)$. On appelle alors A_ψ la borne inférieure finie ou non de la somme des aires de ces rectangles quand on fait varier leurs dimensions et leur nombre.[1]

La définition actuelle de la distance satisfait évidemment aux conditions *i*, *ii* : comme $K_{(f-\phi)}$, $A_{(f-\phi)}$ sont toujours ≥ 0, les quantités $\frac{K}{1+K}$, $\frac{A}{1+A}$ seront toujours finies et ≥ 0. Si $f \equiv \phi$ on a évidemment $(f, \psi)=0$, Réciproquement si $(f, \psi)=0$, on a $(f, \psi)=0$ donc $K_{f-\psi}=0$, *c-a-d.* que quelque soit $\epsilon > 0$, on a

$$\frac{| f(r)-\phi(r) |}{k(r)} \leq \epsilon \qquad \text{d'on } f \equiv \phi.$$

En ce qui concerne la condition *iii*, il résulte de ce que nous avons dit sur la convergence relativement uniforme que

$$K_{(f-\phi)} \leq K_{(f-\psi)} + K_{(\psi-\phi)}$$

or si trois nombres α, β, γ finis on non mais ≥ 0 sont tels que

$$\alpha \leqq \beta + \gamma$$

on a évidemment $\frac{\alpha}{1+\alpha} \leq \frac{\beta}{1+\beta} + \frac{\gamma}{1+\gamma}$; on a donc

$$[f, \phi] \leq [f, \psi] + [\psi, \phi]$$

et il ne reste plus qu'à démontrer que

$$A_{(f-\phi)} \leq A_{(f-\psi)} + A_{(\psi-\phi)}.$$

Or si l'on considère deux systèmes quelconques de rectangles qui enferment les courbes $y= | f-\psi |$, $y= | \psi-\phi |$, il en résultera en combinant les divisions et ajoutant les ordonnées, un système de rectangles qui enfermera $y=(f-\phi)$; l'aire totale des deux premiers systèmes est donc $\geq A_{(f-\psi)}$ et il en sera de même de sa borne inférieure qui est $A_{(f-\psi)} + A_{(\psi-\phi)}$.

Quant à la condition *iv*, elle sera satisfaite d'elle même en la prenant comme définition de la convergence d'une suite de fonctions.

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[1] C'est l'intégrale superieure de Darboux de ψ (x) dans (a, b). Mais nous voulons éviter dans cette note de supposer comme les théories modernes de la mesure et de l'intégration.

Il nous reste cependant à montrer que la convergence ainsi définie implique la convergence ordinaire. Or on a

$$|f(x)-f_n(x)| \leq k(x)\times K_{f-f_n} \leq k(x)\times(f,f_n).$$

si donc (f,f_n) tend vers zéro $f_n(x)$ tend vers $f(x)$. La convergence non seulement a lieu en chaque point x mais elle est uniforme relativement à $k(x)$.

10. Cependant si on a eu soin de prendre pour $k(x)$ une fonction ≥ 0 dont l'aire A_k est infinie (par exemple $k(x)=\frac{1}{x-a}$, $k(a)=0$), la convergence actuelle ne se réduit pas à une convergence uniforme relativement à $k(x)$, ni même à une convergence relativement à quelque suite de fonctions $Q_\epsilon^f(x)$ que ce soit, supposée tendre vers zéro en décroissant quand ϵ tend vers zéro.

Autrement dit la convergence actuelle ne serait pas assurée pour une suite de fonctions $f_n(x)$ et leur limite $f(x)$, si à tout nombre $\epsilon>0$ correspond un entier p, indépendant de x, tel que

$$|f_n(x)-f(x)| \leq |Q_\epsilon^f(x)| \qquad \text{pour } n>p.$$

En effet que les $Q_\epsilon^f(x)$ soient finies ou non désignons par $\lambda_n(x)$ la fonction finie ≥ 0 égale au plus petit des nombres $\frac{k(x)}{n}$ et $\left|Q_{\frac{1}{n}}^f(x)\right|$.

D'après la définition de λ_n, la fonction $f(x)+\lambda_n(x)$ converge vers f relativement aux Q_ϵ^f quand n croit indéfiniment. Par suite $(f,f+\lambda_n)$ et alors aussi A_{λ_n} tendent vers zéro; on en conclut que pour n assez grand : $n=n_0$ A_{λ_n} fini. Dès lors puisque quelque soit l'entier p, l'aire $A_{\frac{k}{p}}$ est infini comme A_k, la courbe $y=\frac{k(x)}{p}$ ne peut jamais partout au dessous de la courbe $y=\lambda_{n_0}(x)$. Il y a donc un abscisse x_p, telle que

$$\lambda_{n_0}(x_p) < \frac{k(x_p)}{p}. \quad \text{Pour } p>n_0, \quad \frac{k(x_p)}{p} < \frac{k(x_p)}{n_0};$$

il y a donc d'après la définition de $\lambda_{n_0}(x)$ un nombre n_0 tel que

$$\left| Q^f_{\frac{1}{n_0}}(x_p) \right| \quad \frac{k(x_p)}{p} \qquad \text{pour } p > n_0.$$

Considérons alors la suite de fonctions $\theta_1(x)$, $\theta_2(x)$ obtenue en prenant $|\theta_n - f|$ partout au plus égal à $\frac{k(x)}{n}$ et en particulier égal à $\frac{k(x_n)}{n}$ pour $x = x_n$. Si on choisit en outre $|\theta_n - f|$ suffisamment petit ailleurs qu'en x_n, on pourra s'arranger pour que $A_{\theta_n - f} < \frac{1}{n}$; comme d'autre part on a évidemment $K_{\theta_n - f} < \frac{1}{n}$ on aura

$$(f, \theta_n) \quad 2 \frac{\frac{1}{n}}{1 + \frac{1}{n}} = \frac{2}{n+1}.$$

Donc la suite des θ_n possède l'espèce de convergence considérée vers la fonction f. Pourtant sa convergence n'est pas uniforme relativement aux Q^f_ϵ puisqu'on a

$$|\theta_n(x_p) - f(x_p| = \frac{|k(x_p)|}{p} > Q^f_{\frac{1}{n_0}}(x_0).$$

Nous avons vu au § 4 que toute définition de la convergence déduite d'une définition de l'écart et impliquant la convergence ordinaire implique nécessairement aussi une convergence uniforme relativement à des fonctions déterminées $P^f_\epsilon(x)$. L'exemple actuel montre que cette convergence uniforme relative n'est pas une propriété caractéristique des convergences définies au moyen de l'écart.

Deuxième Partie.

11. *Généralisation de la convergence en mesure.* Le nom de convergence en mesure a été donné par F. Riesz (dans les Comptes Rendus du 17 Mai 1909) à une espèce de convergence qui se rencontre naturellement quand on étudie les fonctions mesurables ; et F. Riesz en a indiqué quelques propriétés.

Il est intéressant de remarquer qu'on peut généraliser cette notion et l'étendre à des fonctions quelconques sans avoir à se servir de la théorie complète de la mesure au sens de M. Lebesgue. Il suffit d'utiliser les deux remarques fondamentales de M. Borel : " lorsqu'un ensemble est formé de tous les points compris dans une infinité dénombrable d'intervalles n'empiètant pas les uns sur les autres, et ayant une longueur totale s nous dirons que l'ensemble a pour mesure s." " Si E_1 contient tous les éléments de E nous dirons que la mesure de E est inférieure ou égale à α " (α_1 mesure de E_1).

Mais surtout nous allons montrer qu'on peut former une définition de la *distance* de deux fonctions de x quelconques, de telle façon qu'on puisse déduire la convergence en mesure (généralisée) de la notion de distance au moyen des conditions *i, ii, iii, iv.*

12. Nous dirons, en généralisant la définition de F. Riesz, que la fonction $f_n(x)$ converge vers $f(x)$ *en mesure* dans un intervalle I si quelque soit le nombre ω positif l'ensemble des points où $|f_n(x)-f(x)| > \omega$ peut être enfermé dans un ensemble E_n formé d'une suite finie ou non d'intervalles dont la longueur totale tend vers zéro quand n croit indéfiniment.

(On voit que si $f_n(x)$ reste identique à une fonction déterminée $\phi(x)$ quelque soit n, l'ensemble des points où $f-\phi\neq 0$ est de mesure nulle. Or f aura évidemment pour limite $\phi(x)$. Par conséquent pour maintenir l'unidité de la limite d'une suite convergente, nous sommes amenés à considérer comme non distinctes deux fonctions qui ne diffèrent que sur un ensemble de mesure nulle).

Pour abréger et pour rendu plus maniable la définition de la convergence en mesure, nous remplacerons la longueur totale des ensembles E_n qui dépend du choix des intervalles par une quantité qui ne dépend que de $|f_n-f|$ et de ϵ.

13. Désignons par **ME** la borne inférieure de la somme des longueurs d'une suite finie ou non d'intervalles $I_1, I_2, \ldots$ qui enferment un ensemble linéaire E.[1] Si E est un ensemble de points de l'intervalle I, **ME** est au plus égale à la longueur de I. Dans le cas où on peut enfermer E dans un ensemble formé d'une suite d'intervalles dont la longueur totale est aussi petite que l'on veut, on a **ME**$=0$ et on dit que E est de mesure nulle. Il nous sera inutile dans la suite de rien connaître de la théorie des ensembles et des fonctions

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[1] Dans la terminologie de M. Lebesgue **ME** est la mesure extérieure de E.

mesurables; nous nous contenterons des remarques suivantes qui sont bien évidentes :

si E est une partie de l'ensemble linéaire F, on a $\mathbf{M}E \leq \mathbf{M}F$;

si E et F sont deux ensembles linéaires sans points communs, on a $\mathbf{M}E+\mathbf{M}F \geq \mathbf{M}(E+F)$.

14. Appelons $E_{|f_n-f|>\omega}$ l'ensemble des points où $|f_n(x)-f(x)|>\omega$, on voit que $f_n(x)$ sera dit converger en mesure vers $f(x)$ sur I si, quelque soit $\omega>0$, $\mathbf{M}E_{|f_n-f|>\omega}$ tend vers zéro avec $\frac{1}{n}$.

Il faut bien remarquer que $f_n(x)$ peut converger en mesure vers $f(x)$ dans un intervalle I sans que $f_n(x)$ converge vers $f(x)$ en chaque point de I et même sans que $f_n(x)$ converge vers $f(x)$ en un seul point de I. Pour le montrer divisons I en deux, quatre, huit,..., 2^p,... intervalles égaux et numérotons les intervalles obtenus I_1, I_2 quand on partage I en 2 parties, I_3, I_4, I_5, I_6 quand on le partage en quatre parties etc. .. Puis appelons $f_n(x)$ la fonction qui représente une ligne brisée formée d'une parallèle à Ox d'ordonnée dans I_n, d'une droite inclinée joignant cette parallèle à Ox dans chacun des intervalles de même longueur que I_n à droite et à gauche de I_n et de la droite Ox dans le reste de l'intervalle I.

On voit que

$$\mathbf{M}E_{|f_n|>\epsilon} < 3\, I_n$$

quelque soient l'entier n et le nombre positif ϵ. Par conséquent $f_n(x)$ converge en mesure vers zéro dans I. Cependant il est évident que pour tout point x, $f_n(x)$ passe une infinité de fois par les valeurs zéro et un de sorte que le nombre $f_n(x)$ ne converge vers zéro pour aucune valeur de x.

15. Cependant dans la communication à l'Académie des Sciences rappelée plus haut, F. Riesz a fait remarquer que l'on peut extraire d'une suite qui converge en mesure une suite qui converge pour chaque valeur de x sauf peut-être dans un ensemble de mesure nulle. On peut en déduire en suite au moyen d'un théorème d'Egoroff que cette seconde suite converge uniformément sauf dans un ensemble de mesure aussi petite que l'on veut. La démonstration de F. Riesz faite dans le cas des fonctions mesurables s'étend sans difficulté au cas général et même donne directement le second résultat indiqué, comme nous allons le voir,

Soit $$\omega_1+\omega_2+\omega_3+\dots$$

une série numérique convergente à termes positifs décroissants et $\epsilon_1, \epsilon_2,$ une suite de nombres positifs qui tendent en décroissant vers zéro. Si une suite de fonctions $f_n(x)$ converge en mesure vers $f(x)$ sur un intervalle I, la quantité

$$\mathrm{ME}_{|f_n-f|>\epsilon_p}$$

tend vers zéro quand n croît indéfiniment. Elle est donc inférieure à ω_p à partir d'un certain rang n_p, qu'on peut supposer croître avec p.

Appelons alors E_p l'ensemble des points qui appartiennent à l'un des ensembles

$$E_{|f-f_{n_p}|>\epsilon_p}, \quad E_{|f-f_{n_{p+1}}|>\epsilon_{p+1}}, \quad \dots;$$

On a

$$\mathrm{ME}_p \leqq \omega_p+\omega_{p+1}$$

de sorte que ME_p tend vers zéro avec $\frac{1}{p}$ et que si μ est un nombre positif quelconque on peut prendre p assez grand pour que E_p soit contenu dans une suite $\int$ d'intervalles de longueur totale $<\eta$.

Or on a en tout point qui n'appartient pas à $\int$

$$|f-f_{n_r}|<\epsilon_r$$

quelque soit $r>p$. Autrement dit la suite $f_{n_1}, f_{n_2},\dots$ extraite de la suite donnée des f_n converge uniformément sauf dans une suite $\int$ d'intervalles dont la longueur totale peut être rendue inférieure à une quantité arbitraire η.

Il en résulte en particulier que l'ensemble des points s'il en existe où les f_{n_p} ne convergent pas vers f est de mesure nulle, étant au plus égale à η, nombre positif arbitraire.

16. Mais arrivons au second point que j'avais en vue. Si l'on considère deux fonctions quelconques $f(x)$, $\phi(x)$ et si l'on forme l'expression

$$\omega+\mathrm{ME}_{|f-\phi|>\omega}$$

pour toute valeur positive de ω, cette quantité a une borne inférieure ≥ 0, borne qui est d'ailleurs évidemment au plus égale à la longueur de l'intervalle I où f et ϕ sont définis. Appelons cette borne la *distance* de $f(x)$ et de $\phi(x)$ sur l'intervalle I et représentons là par la notation (f, ϕ). On a évidemment $(f, \phi)=(\phi, f)\geq 0$. Il est utile de remarquer que si $(f, \phi) <\delta$ on peut trouver un nombre $\omega_0>0$ tel que

$$\omega_0+\mathrm{ME}_{|f-\phi|>\omega_0} < \delta \quad , \quad \text{d'où}$$

$$\omega_0<\delta,\ \mathrm{ME}_{|f-\phi|>\omega_0} <\delta ; \text{ et comme } \mathrm{E}_{|f-\phi|>\delta}$$

est contenu dans $\mathrm{E}_{|f-\phi|>\omega_0}$, on a enfin

$$\mathrm{ME}_{|f-\phi|>\delta} < \delta \tag{3}$$

Il s'agit maintenant d'examiner si les conditions i, ii, iii, iv sont satisfaites.

On remarque que si f et ϕ sont identiques, $(f, \phi)=0$, mais que la réciproque n'est pas vraie. Seulement, d'après l'inégalité précédente, on voit que si $(f, \phi)=0$ on a quelque soit $\delta>0$ et n entier

$$\mathrm{ME}_{|f-\phi|>\frac{\delta}{2^n}} < \frac{\delta}{2^n}$$

d'où

$$\mathrm{ME}_{|f-\phi|>0} \leq \mathrm{ME}_{|f-\phi|>\delta} + \mathrm{ME}_{|f-\phi|>\frac{\delta}{2}} + \cdots$$
$$+ \mathrm{ME}_{|f-\phi|>\frac{\delta}{2^n}} + \cdots$$

$$\leq \delta + \frac{\delta}{2} + \cdots + \frac{\delta}{2^n} + \ldots < 2\delta$$

Donc f et ϕ ne diffèrent que dans un ensemble de mesure nulle.

Si donc *on ne considère pas comme distinctes* deux fonctions qui ne diffèrent que tout au plus dans un ensemble de mesure nulle, la condition nécessaire et suffisante pour que deux fonctions soient distinctes est que leur distance soit positive. Il faut d'ailleurs remarquer que si f, ϕ sont deux fonctions quelconques et si f_1, ϕ_1 sont deux fonctions qui ne diffèrent de f, ϕ respectivement que tout au plus dans un ensemble de mesure nulle, on a $(f, \phi)=(f_1, \phi_1)$. Car on aura

$$|f_1-\phi_1| = |f-\phi|$$

sauf peut-être dans un ensemble de mesure nulle et par suite

$$\mathrm{ME}_{|f_1-\phi_1|>\omega} = \mathrm{ME}_{|f-\phi|>\omega}$$

quelque soit >0.

Passons à la condition *iii*; soient f, ϕ, ψ trois fonctions quelconques. Posons $\alpha=(f, \psi)$, $\beta=(\phi, \psi)$. Quelque soit le nombre positif δ il existe deux nombres ω, ω' tels que

$$\omega+\mathrm{ME}_{|f-\psi|>\omega} < \alpha+\delta$$

$$\omega'+\mathrm{ME}_{|\phi-\psi|>\omega'} < \beta+\delta$$

Or on a partout

$$|f-\phi| \leq |f-\psi| + |\phi-\psi|$$

de sorte que si un point n'appartient ni á $\mathrm{E}_{|f-\psi|>\omega}$ ni à $\mathrm{E}_{|\phi-\psi|>\omega,}$ il ne peut appartenir à $\mathrm{E}_{|f-\phi|>\omega+\omega'}$ autrement dit le dernier ensemble est contenu dans la somme des deux premiers et par suite

$$(f, \phi)\leq\omega+\omega'+\mathrm{E}_{|f-\phi|>\omega+\omega'} < \alpha + \beta + 2\,\delta \;.$$

On a donc

$$(f, \phi)\leq(f, \psi) + (\psi, \phi)+2\,\delta$$

quelque soit $\delta>0$ et par suite

$$(f, \phi) \leq (f, \psi) + (\psi, \phi).$$

Enfin arrivons à la condition *iv*. Si (f, f_n) tend vers zéro avec $\dfrac{1}{n}$ on a quelque soit ϵ

$$(f, f_n) < \epsilon$$

pour n assez grand, et par suite par analogie avec (3)

$$\mathrm{ME}_{|f-f_n|>\epsilon} < \epsilon.$$

Donc f_n converge en mesure vers f.

Réciproquement si f_n converge en mesure vers f_1 on a quelsque soient les nombres positifs ϵ et μ

$$\mathrm{ME}_{|f-f_n|>\eta} < \epsilon$$

et par suite

$$(f, f_n) < \epsilon + \eta,$$

pour n assez grand. Donc (f, f_n) tend vers zéro avec $\frac{1}{n}$.

Ainsi il est bien établi que si l'on considère comme non distinctes deux fonctions qui ne diffèrent que dans un ensemble de mesure nulle, *on peut faire correspondre à tout couple de fonctions* $f(x)$, $\phi(x)$ un nombre (f, ϕ) appelé *distance* de f et ϕ qui satisfait aux conditions i, ii, iii et *au moyen du quel la convergence en mesure peut être definie* par la condition *iv.*[1]

D'après la démonstration précédente, on voit que non seulement si la distance (f, f_n) tend vers zéro, f_n converge en mesure vers f, mais encore si la série

$$(f_1 f_1) + (f_1 f_2) + \dots + (f_1 f_n) + \dots$$

est une série convergente à termes décroissants, la fonction f_n converge uniformément vers f sauf peut-être dans un ensemble d'intervalles dont la longueur totale peut être prise aussi petite que l'on veut.

17. Dans le cas des fonctions mesurables, F. Riesz a indiqué la condition nécessaire et suffisante pour qu'une suite de fonctions données $f_n(x)$ soit convergente en mesure. Il est facile d'étendre sa démonstration au cas général; mais il est plus intéressant de formuler la condition au moyen de la distance et de montrer que celle-ci se prête à la généralisation du criterè de convergence de Cauchy :

Pour qu'une suite de fonctions $f_n(x)$ quelconques converge en mesure il faut et il suffit que quelque soit le nombre $\epsilon > 0$, on puisse trouver un entier n tel que

$$(f_n, f_{n+p}) < \epsilon \qquad \textit{quelque soit l'entier } p.$$

La condition est évidemment nécessaire puis que si la suite converge en mesure vers une fonction f, on a

$$(f_n, f_{n+p}) \leq (f_n, f) + (f_{n+p}, f)$$

Elle est aussi suffisante. En effet soit

$$\omega_1 + \omega_2 + \dots$$

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[1] Ce résulat a été énoncé pour la première fois dans ma note aux Comptes Rendus du 24 Janvier 1916.

une série convergente à termes positifs décroissants ; d'après l'hypothèse, on peut trouver, quelque soit l'entier q, un entier n_q tel que

$$(4) \qquad (f_{n_q}, f_{n_{q+p}}) < \omega_q$$

d'où

$$\mathbf{ME}_{|f_{n_q} - f_{n_q}^{\;+p}| > \omega_q} < \omega_q$$

En particulier, on peut enfermer dans une suite $\int_q$ d'intervalles de longueur totale $< \omega_q$ tous les points tels que

$$|f_{n_q}(x) - f_{n_{q+1}}(x)| > \omega_q$$

Et, par suite, pour tout point qui n'appartient pas à la suite d'intervalles Σ_q formée de ceux de $\int_q$, de $\int_{q+1}$, etc. ....., on a

$$|f_{n_q}(x) - f_{n_{q+1}}(x)| \leq \omega_q, \ldots, |f_{n_{q+r}}(x) - f_{n_{q+r+1}}(x)| \leq \omega_{q+r}, \quad \ldots$$

d'où en particulier

$$(5) \qquad \left| f_{n_{q+r}}(x) - f_{n_{q+r+p}}(x) \right| \leq \omega_{q+r} + \ldots \ldots + \omega_{q+r+p-1}$$

Etant donnés deux nombres positifs arbitraires ϵ et η, prenons d'abord q assez grand pour que

$$\omega_q + \omega_{q+1} + \quad \ldots \quad < \eta$$

puis r assez grand pour que

$$\omega_{q+r} + \omega_{q+r+1} \quad \ldots \quad < \epsilon$$

On voit qu'on aura en dehors d'un ensemble Σ_q d'intervalles de longueur totale $< \eta$

$$(6) \qquad \left| f_{n_m}(x) - f_{n_{m+p}}(x) \right| < \epsilon$$

pour m assez grand $(m = q + r)$. Autrement dit la suite des fonctions $f_{n_1}, f_{n_2}, \ldots$ converge uniformément en dehors de Σ_q. Il en résulte en particulier que cette suite ne peut diverger que dans l'ensemble J des points communs à tous les Σ_q. Cet ensemble J étant contenu quelque soit q dans une suite d'intervalles de longueur totale inférieure à $\omega_q + \omega_{q+1} + \ldots$ est de mesure nulle. La suite des fonctions $f_{n_1}(x), f_{n_2}(x), \ldots$ converge donc vers une fonction $f(x)$ bien déterminée sauf peut-être dans un ensemble J de mesure nulle. Achevons de déterminer $f(x)$ en lui donnant par exemple la valeur zéro sur J. En faisant croître p dans (5), on voit qu'on a pour $r=0$.

$$\left| f_{n_q}(x) - f(x) \right| \leq \omega_q + \ldots\ldots$$

sauf dans Σ_q et par suite

$$\left(f, f_{n_q} \right) \leq 2\,(\omega_q + \omega_{q+1} + \ldots\ldots)$$

E combinant avec (4), on obtient

$$\left(f, f_{n_q+p} \right) \leq 3\,\omega_q + 2\,(\omega_{q+1} + \omega_{q+2} + \ldots\ldots)$$

quelsque soient p et q; par suite non seulement la suite des f_{n_q} mais la suite tout entière des f_n converge *en mesure* vers f.

18. *Convergence presque uniforme.* Nous avons vu que d'une suite de fonctions qui converge en mesure, on peut tirer une suite de fonctions qui converge vers la même limite uniformément sauf peut être dans une suite d'intervalles de longueur aussi petite qu'on voudra. Nous dirons qu'elle converge *presque uniformément.* La convergence presque uniforme est un cas particulier de la convergence en mesure. Une fonction f_n qui converge presque uniformément vers $f(x)$ converge en mesure vers $f(x)$ elle converge aussi en chaque point x sauf peut être dans un ensemble de mesure nulle; par consequent l'exemple donné § 14 montre qu'une suite peut converger en mesure sans converger presque uniformément. On voit aussi que pour conserver limitè de la limite presque uniforme il faut considérer comme non distantes deux fonctions qui ne diffèrent que sur un ensemble de mesure nulle.

La notion de convergence presque uniforme peut-elle être déduite d'une définition convenable de la distance ou au moins de l'écart?

La réponse est négative. Supposons en effet qu'on puisse assigner à tout couple de fonctions $f(x)$, $\phi(x)$ définies dans un intervalle I, un écart $[f, \phi] = [\phi, f] \geq 0$ tel que cet écart soit nul dans le cas où f ne diffère de ϕ que dans un ensemble de mesure nulle et que la condition nécessaire et suffisante pour que f_n converge presque uniformément vers f soit que $[f_n, f)$ tende vers zéro. Je vais montrer qu'on arrive à une contradiction.

Reprenant l'exemple de § 14, divisons en effet l'intervalle I en deux parties égales, puis en quatre, etc. et assignons à chacun de ces intervalles une certaine fonction comme dans l'exemple § 14. Toutefois il nous sera plus commode de désigner par I_m^n le $n^{ième}$ des intervalles de longueur $\frac{1}{2^m}$ et par $f_m^{(n)}(x)$ la fonction correspondante. Soit alors k_m le plus grand des écarts $[0, f_m^{(n)}(x)]$ de la fonction zéro avec les différentes fonctions $f_m^{(n)}(x)$ qui correspondent aux intervalles de longueur $\frac{1}{2^m}$.

Les nombres positifs $k_1, k_2, k_3, \ldots\ldots k_m, \ldots\ldots$ tendent vers zéro avec $\frac{1}{m}$. Sans quoi on pourrait en extraire une suite.

$$k_{m_1},\ k_{m_2}, \ldots$$

qui aurait une borne inférieure positive λ Si alors

$$k_{m_i} = \left[f_{m_i}^{(n_i)}, 0\right]$$

et si $C_{m_i}^{n_i}$ est le milieu de l'intervalle $I_{m_i}^{n_i}$, on pourrait extraire de la suite bornée des points $C_{m_i}^{n_i}$ une suite

$$C_{m'_1}^{n'_1},\ C_{m'_2}^{n'_2}, \ldots\ldots$$

qui tend vers un certain point C de I.

Or
$$\left[f_{m'_i}^{(n'_i)}, 0\right] = k_{m'_i} > \lambda > 0 .$$

donc la suite des fonctions $f_{m'_i}^{(n'_i)}$ ne peut converger vers zéro presque uniformément. Mais d'autre part soit η un nombre positif donné. On peut prendre i assez grand pour qu'à partir d'un certain rang $I_{m'_i}^{n'_i}$ soit tout entier ainsi que les deux intervalles contigus contenus dans l'intervalle J de longueur η et de centre C. Par conséquent à partir d'un certain rang $f_{m'_i}^{(n'_i)} = 0$ en dehors de J. Dans la suite des $f_{m'_i}^{(n'_i)}$ converge presque uniformément vers zéro. On arriverait ainsi à une contradiction.

Mais alors si k_1, k_2......k_n,......tendent vers zéro la suite des fonctions $f_2^1, f_2^2, f_4^1, f_4^2, f_4^3, f_4^4, f_8^1$,...... devra converger presque uniformément vers zéro puisque la suite de ses écarts avec zéro est formée de nombres ≥ 0 qui sont au plus égaux à la suite

$$k_2,\ k_2,\ k_4,\ k_4,\ k_4,\ k_4\ ;\ k_8,\dots\dots$$

qui tend vers zéro. Or nous avons vu que cette suite de fonctions converge pas vers zéro en aucun point de Ox.

19. *Cas des fonctions mesurables.* Lorsqu'on se limite au cas des fonctions mesurables, la quantité que nous avons appelé $\mathrm{ME}_{|f-\phi|>\omega}$ se réduit à la mesure au sens de Lebesgue de l'ensemble mesurable $\mathrm{E}_{|f-\phi|>\omega}$. Alors d'après un théorème de Borel : toute série convergente de fonctions mesurables converge en mesure. Il en est de même si la série converge *presque partout*, c'est à dire, suivant la terminologie de M. Lebesgue, si la série converge sauf en un ensemble de points de mesure nulle. Dans ce même cas des fonctions mesurables la convergence presque partout coïncide avec la convergence presque uniforme.

Par conséquent, en considérant encore comme non distinctes deux fonctions égales presque partout on voit qu'*il est impossible* dans le champ des fonctions mesurables *de définir un ecart* (f, ϕ) de deux fonctions mesurables f, ϕ, écart satisfaisant aux conditions *i*, *ii*, *iv* *de façon à définir* (au moyen de la condition *iv*) *la convergence presque partout par l'intermédiaire d'un écart.* Nous avons vu au contraire que la convergence en mesure peut se définir par l'intermédiaire d'un écart et même d'une " distance " (la condition *iii* étant réalisée.)

On the convergence of the solutions of the second kind of Mathieu's Equation.

BY

SASINDRACHANDRA DHAR.

[*Read, September 4th, 1920.*]

The object of the present paper is to investigate the convergence of the solutions of Mathieu's Equation of the second kind, which have been investigated by Mr. Lindsay Ince,* who has given two methods for finding the series of integrals ; but neither of these methods is suitable for studying the convergence of the series. In the present paper I have given another method for constructing the series, which will be suitable for the consideration of its convergency. This latter method follows lines similar to the method of Frobenius as applied by him to the solutions of linear differential equation in power series and is similar to the method employed by Professors Whittaker and Watson† for constructing the solutions of the first kind.

§ 1

The connonical form of Mathieu's differential equation is given by

$$\frac{d^2y}{dz^2}+(a+16q+16q\cos 2z)y=0 \qquad \ldots \quad (1)$$

For certain values of 'a', two kinds of solutions of the above differential equation have been constructed. The periodic solutions of the first kind have been denoted by Prof. Whittaker‡ in the forms .—

$$\left.\begin{array}{llll} ce_0(z,q), & ce_1(z,q), & \ldots & ce_m(z,q),\ldots \\ & se_1(z,q), & \ldots & se_m(z,q),\ldots \end{array}\right\} \qquad \ldots \quad (2)$$

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* E. Lindsay Ince, "The Elliptic Cylinder Functions of the second kind." (*Proc. Edin. Math. Soc.*, Vol. XXXIII, 1914-15).

† Whittaker and Watson : *Modern Analysis*, pp. 413-15.

‡ Whittaker : *Fifth International Congress of Mathematics*, 1912.

The solutions of the second kind corresponding to the above solutions of the first kind, have, however, been obtained by Mr. E. Lindsay* Ince and they are generally denoted as

$$\left.\begin{array}{ll} i\eta_0(z,q),\ i\eta_1(z,q), \quad \ldots & i\eta_m(z,q), \ldots \\ \qquad\quad i\eta_1(z,q), \quad \ldots & i\eta_m(z,q), \ldots \end{array}\right\} \qquad \ldots \quad (3)$$

The special value of "a" for which the two solutions $ce_m(z,q)$ and $i\eta_m(z,q)$ have been constructed, is given by

$$a+16q=m^2+\frac{32q^2}{m^2-1}-\frac{128(5m^2+7)q^4}{(m^2-1)^3(m^2-4)}-\ etc. \quad \ldots \quad (4)$$

We shall denote it, however, as

$$a+16q=m^2+a_1q+a_2q^2+a_3q^3+\ \ldots\ \&c, \quad \ldots \quad (5)$$

where $a_1=0,\ a_2=\dfrac{32}{m^2-1},\ \ldots$ etc.

§ 2

If we put

$$a+16q=m^2+8p,$$

the equation (1) can be written in the form

$$\frac{d^2y}{dz^2}+m^2y=-8(p+2q\cos 2z)y \quad \ldots \quad (6)$$

When p and q are neglected, the solutions of the equation are given by

$$y=\cos mz, \text{ and } y=\sin mz.$$

If, however, we proceed with the solution $y=\cos mz$, it will enable us to construct the solution $ce_m(z,q)$. We will therefore, proceed with the second one $y=\sin mz$.

Let us denote $\sin mz=U_0(z)$. Then to obtain a closer approximation, we will write $-8(p+2q\cos 2z)U_0(z)$ as a series of sines of multiples of z, in the form

$$-8\{q\sin(m-2)z+p\sin mz+q\sin(m+2)z\}$$

which we shall denote by $\quad V_1(z). \quad \ldots \quad (7)$

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* Lindsay: *Proc. Edin Math. Soc*, Vol. XXXIII, p. 6.

Then, instead of solving the differential equation

$$\frac{d^2y}{dz^2}+m^2y=V_1(z),$$

we will solve the equation

$$\frac{d^2y}{dz^2}+m^2y=W_1(z), \qquad \ldots \quad (8)$$

where

$$W_1(z)=V_1(z)+(8p-a_1q)\sin mz. \qquad \ldots \quad (9)$$

Its integral, which we shall denote by $U_1(z)$ is given by

$$U_1(z)=\frac{-2q\sin(m-2)z}{m-1}+\frac{2q\sin(m+2)z}{m+1}+\frac{a_1q.\cos mz}{2m} \qquad \ldots \quad (10)$$

§ 3

To obtain a still closer approximation, we will express $-8(p+2q\cos 2z)U_1(z)$ as a series in sines of multiples of z, which we shall denote by $V_2(z)$, *viz.*.

$$V_2(z)=\frac{16q^2\sin(m-4)z}{m-1}+\frac{16q^2}{m+1}\sin(m+4)z-\frac{8q^2a_1}{2m}z\cos(m2)z$$

$$-\frac{16pq}{m+1}\sin(m+2)z-\frac{16q^2}{m+1}\sin(m+4)z-\frac{8q^2a_1}{2m}z\cos(m-2)z$$

$$-\frac{8pqa_1}{2m}z\cos mz-\frac{8q^2a_1}{2m}z\cos(m+2)z. \qquad \ldots \quad (11)$$

Here again we solve the equation

$$\frac{d^2y}{dz^2}+m^2y=W_2(z), \qquad \ldots \quad (12)$$

where $W_2(z)=V_2(z)-a_1q^2\sin mz+\lambda_2 z\cos mz$, where λ_2 has been determined in such a way that $W_2(z)$ does not involve $z\cos mz$* (if there be any).

Suppose that $U_2(z)$ is the integral of the equation (12). We will proceed exactly with $U_2(z)$ as we have done with $U_1(z)$ and obtain the integral $U_3(z)$.

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* It will be found thot $z\cos mz$ first appears in $U_m(z)$ andhence in $V_{m+1}(z)$.

§ 4.

Continuing thus, we get the series of integrals $U_0(z)$, $U_1(z)$, $U_2(z)$, ... $U_n(z)$, ... etc., of the differential equations

$$\left.\begin{array}{l} \dfrac{d^2y}{dz^2}+m^2y=0, \\[2ex] \dfrac{d^2y}{dz^2}+m^2y=W_1(z), \\[2ex] \quad\cdots\quad\cdots \\ \quad\cdots\quad\cdots \\[1ex] \dfrac{d^2y}{dz^2}+m^2y=W_n(z), \\[2ex] \quad\cdots\quad\cdots \\ \quad\cdots\quad\cdots \end{array}\right\} \qquad \cdots \quad (13)$$

respectively, where

$$W_n(z)=V_n(z)-a_nq^n\sin mz+\lambda_n z\cos mz\,; \quad (n\geq 2)$$

$$V_n(z)=-8(p+2q\cos 2z)U_{n-1}(z)\,; \quad (n\geq 1)$$

Therefore from (9) and (13), we have

$$\left(\frac{d^2}{dz^2}+m^2\right)\sum_{n=0}^{\infty}U_n(z)=\sum_{n=1}^{\infty}W_n(z)$$

$$i.e.=\sum_{n=1}^{\infty}V_n(z)+\left(8p-\sum_{1}^{\infty}a_nq^n\right)\sin mz$$

$$+\sum_{2}^{\infty}\lambda_n z\cos mz.$$

or

$$\left\{\frac{d^2}{dz^2}+(a+16q+16q\cos 2z)\right\}\sum_{0}^{\infty}U_n(z)$$

$$=\left(8p-\sum_{1}^{\infty}a_nq^n\right)\sin mz+\sum_{2}^{\infty}\lambda_n z\cos mz.$$

But we have from (3)

$$8p=\sum_{1}^{\infty}a_nq^n,$$

and it will also be found that $\sum_{2}^{\infty}\lambda_n$ also vanishes for the above value of "p." Hence $\sum_{0}^{\infty}U_n(z)$ is the solution of Mathieu's equation, corresponding to the value of "a" given by (3). It is, in fact, the solution of the second kind, corresponding to $ce_m(z,q)$, as have been obtained by Mr. Lindsay Ince,* that is,

$$i\eta_m(z,q) = \sum_{n=0}^{\infty} U_n(z) \qquad \dots \quad (14)$$

§ 5.

We proceed to show that $\sum\lambda_n$ actually vanishes, by working out a few particular cases. Suppose we construct the integral which corresponds to $ce_1(z,q)$. The particular value of the arbitrary constant "a" for which $ce_1(z,q)$ is obtained, is given by

$$a+16q=1-8q-8q^2+8q^3-\tfrac{8}{3}q^4-\tfrac{88}{9}q^5+\dots \qquad \dots \quad (15)$$

When we write $a+16q=1+8p$. the differential equation (1) reduces to

$$\frac{d^2y}{dz^2}+y=-8(p+2q\cos 2z)y. \qquad \dots \quad (16)$$

Here $U_0(z)$ is evidently $\sin z$.

(a) To get $U_1(z)$, we express $-8(p+2q\cos 2z)U_0(z)$ in a series of sines in the form

$$-(p-q)\sin z-8q\sin 3z\equiv V_1(z)$$

and solve the equation

$$\frac{d^2y}{dz^2}+y=W_1(z),$$

where $W_1(z)=V_1(z)+8(p+q)\sin z$, since there is no term $z\cos z$ contained in $V_1(z)$.

The integral of the latter equation is

$$U_1(z)=-8qz\cos z+q\sin 3z. \qquad \dots \quad (16A)$$

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* Lindsay Ince : *loc cit.*

(*b*) Again, we express $-8(p+2q\cos 2z)U_1(z)$ in a series of sines in the form

$$64q(p+q)z\cos z+64q^2 z\cos 3z-8pq\sin 3z-8q^2\sin 5z-8q^2\sin z\equiv V_2(z).$$

Then we shall have to find an integral of the differential equation

$$\frac{d^2y}{dz^2}+y=W_2(z), \qquad \dots \quad (17)$$

where $W_2(z)=V_2(z)-64q(p+q)z\cos z+8q^2\sin z$ (*i.e.*, $W_2(z)$ is hereby made independent of $z\cos z$).

Hence $\qquad W_2(z)=64q^2 z\cos 3z-8pq\sin 3z-8q^2\sin 5z.$

The integral of the equation (17) can be obtained by adding up the particular integrals of the equations :—

$$\left.\begin{aligned}\frac{d^2y}{dz^2}+y&=-8q^2\sin 5z,\\ \frac{d^2y}{dz^2}+y&=-8pq\sin 3z,\\ \frac{d^2y}{dz^2}+y&=-64q^2 z\cos 3z,\end{aligned}\right\} \qquad \dots \quad (18)$$

and it is given by

$$U_2(z)\equiv-8q^2 z\cos 3z+q(6q+p)\sin 3z+\frac{q^2\sin 5z}{3} \qquad \dots \quad (18\text{A})$$

(*c*) Again, since $-8(p+2q\cos 2z)U_2(z)$

$$\begin{aligned}&=64q^3 z\cos 5z+64pq^2 z\cos 3z+64q^3 z\cos z\\ &\qquad -8q^2(6q+p)\sin z-\{8pq(6q+p)+\tfrac{8}{3}q^3\}\sin 3z\\ &\qquad\qquad -8\{q^2(6q+p)+\tfrac{1}{3}pq^2\}\sin 5z-\tfrac{8}{3}q^3\sin 7z\equiv V_3(z),\end{aligned}$$

we have

$$W_3(z)=V_3(z)-64q^3 z\cos z-8q^3\sin z,$$

so that $W_3(z)$ may be independent of $z\cos z$.

Now, solving the equation

$$\frac{d^2y}{dz^2}+y=W_3(z), \qquad \dots \quad (19)$$

we find

$$U_3(z) = -\tfrac{8}{3}q^3 z\cos 5z - 8pq^2 z\cos 3z + 4q^2(7q+p)z\cos z$$
$$+ q(\tfrac{1}{3}q^2 + 12pq + p^2)\sin 3z + \tfrac{1}{9}q^2(7q+p)\sin 5z$$
$$+ \tfrac{1}{18}q^3\sin 7z. \qquad \ldots \quad (19A)$$

(*d*) Again, $V_4(z) = -8(p + 2q\cos 2z)U_3(z)$. Now, if we express it in a series of sines, we find that the term which contains $z\cos z$ as a factor is

$$-32q^2(p^2 + 6q + 7q^2)z\cos z \qquad \ldots \quad (20)$$

and the term containing $\sin z$, is

$$-8q^2(p^2 + 12pq + \tfrac{1}{3}q^2)\sin z. \qquad \ldots \quad (21)$$

Hence we define $W_4(z)$ such that

$$W_4(z) = V_4(z) + 32q^2(p^2 + 6pq + 7q^2)\cos z + \tfrac{8}{3}q^4\sin z,$$

$W_4(z)$ is thus made independent of z. We can now find the integral $U_4(z)$.

Proceeding thus, we can find all the integrals $U_0(z)$, $U_1(z)$, $U_2(z)$, ... $U_n(z)$,...etc., to as many terms as we please.*

§ 6

Hence if, in

$$U_0(z) + U_1(z) + U_2(z) + U_3(z) + \ldots$$

as found in § 6, we substitute by (15)

$$8p = -8q - 8q^2 + 8q^3 - \tfrac{8}{3}q^4 - \tfrac{88}{9}q^5 + \ldots,$$

we get, after arranging

$$-8q(1 - 3q^2 + \ldots)\cdot z\{\cos z + q\cos 3z + q^2(-\cos 3z + \tfrac{1}{3}\cos 5z) + \ldots$$
$$\} + \sin z + q\sin 3z + q^2(\tfrac{1}{3}\sin 5z + 5\sin 3z)$$
$$+ q^3(\tfrac{1}{18}\sin 7z + \tfrac{8}{3}\sin 5z - \tfrac{35}{3}\sin 3z) + \ldots \dagger$$

(correct to third power of q). ... (22)

---

* For finding the U's, the particular integrals of the following equations will be found very useful :—

(*a*) $y'' + m^2y = A\sin(m+a)z$; (*b*) $y'' + m^2y = A\sin mz$; (*c*) $y'' + m^2y = A\cos az$, whose particular integrals are respectively

(*a*) $y = -A\sin(m+a)z/a(2m+a)$; (*b*) $y = -Az\cos mz/2m$;

(*c*) $y = Az\cos az/(m^2 - a^2) + 2Aa\sin az/(m^2 - a^2)^2$.

† Lindsay Ince : *loc cit.*

And here it will be noticed that $\sum\limits_{2}^{\infty}\lambda_n$ as obtained above is given by

$$-64q(p+q)-64q^3+32q^2(p^2+6pq+7q^2)+\,..$$

which, on substitution of the value of p in terms of q, vanishes if we neglect powers of q beyond the fourth.

Hence the series as obtained in (22) is a solution of the differential equation (1) corresponding to the value of "a" as given in (15). This is the second solution corresponding to $ce_1(z,q)$ and is denoted by $i\eta_1(z,q)$.

§ 7

Similarly if we proceed to construct the integral corresponding to to the value of "a" of $ce_2(z,q)$, *viz.*,

$$a+16q=4+\frac{80}{3}q^2-\frac{6104}{27}q^4+\dots \qquad \dots\ (23)$$

we find

$$\left.\begin{aligned}
&U_0(z)=\sin 2z,\\
&U_1(z)=\frac{2}{3}\,q\sin 4z,\\
&U_2(z)=\frac{1}{6}\,q^2\sin 6z+\frac{4}{9}\,pq\sin 4z+8q^2z\cos 2z,\\
&U_3(z)=\frac{1}{45}q^3\sin 8z+\frac{11}{72}pq^2\sin 6z+\frac{1}{9}\left(\frac{8}{3}p^2q-31q^3\right)\sin 4z\\
&\qquad\qquad -16q^3z+\frac{8}{9}\,pq^2z\cos 2z+\frac{16}{3}q^3z\cos 4z,
\end{aligned}\right\}\dots \qquad (24)$$

.. etc....

and

$$\sum\lambda_n=64pq^2+\left(\frac{64}{9}p^2q^2-256q^4+\frac{128}{3}\,q^4\right)+\text{higher powers of } q$$

than the fourth

$$=8q^2\left\{\frac{80}{3}q^2+\dots\right\}+\frac{1}{9}\,q^2\left\{\frac{80}{3}q^2+\dots\right\}^2-256q^2+\frac{128}{3}\,q^4+\dots$$

$\doteq 0$, neglecting higher powers of q than the fourth.

Hence $\sum U_n(z)$ is the second solution corresponding to $ce_2(z,q)$ and is evidently the solution $i\eta_2(z,q)$.

That $\sum\lambda_n$ vanishes, has been further verified in a few other cases.

§ 8

We shall now examine the solution $\Sigma U_n(z)$ more closely with a view to study its convergence. The forms of $U_n(z)$, which are solutions of the differential equation

$$\frac{d^2y}{dz^2}+m^2y=W_n(z),$$

will be of the types :—

(*i*) when $n<m$,

$$U_n(z)={}^{*}\sum_{r=1}^{n}{}'\beta_{n,r}\sin(m-2r)z+\sum_{r=1}^{m}\alpha_{n,r}\sin(m+2r)z$$

(*ii*) when $n=m$,

$$U_m(z)=\sum_{r=1}^{m}{}'\beta_{m,r}\sin(m-2r)z+\sum_{r=1}^{m}\alpha_{m,r}\sin(m+2r)z+\delta_{0,0}z\cos mz.$$

(*iii*) when $n>m$, *i.e.*, when $n=m+\eta$, $\eta\geq 1$

$$U_{m+\eta}(z)=\sum_{r=1}^{m+\eta}\beta_{m+\eta,r}\sin(m-2r)z+\sum_{r=1}^{m+\eta}\alpha_{m+\eta,r}\sin(m+2r)z$$

$$+z\left\{\sum_{r=1}^{\eta}{}'\gamma_{\eta,r}\cos(m-2r)z+\sum_{r=0}^{\eta}\delta_{\eta,r}\cos(m+2r)z\right\}$$

Then since

$$\left(\frac{d^2}{dz^2}+m^2\right)U_{m+\eta+1}(z)=-8(p+2q\cos 2z)U_{m+\eta}(z)$$

$$-\lambda_{m+\eta+1}z\cos mz-a_nq^n\sin mz,$$

---

* Σ' denotes that the summation ceases at the highest integral value of r which is less than $\frac{m}{2}$.

we get on equating the coefficients of $z\cos(m+2r)z$, $z\cos(m-2r)z$, $\sin(m+2r)z$, and $\sin(m-2r)z$, the following recurrence formula

$$(a)\ r(m+r)\delta_{\eta+1,r} = \{p\delta_{\eta,r} + q(\delta_{\delta,r-1}\delta_{\eta,r+1})\},\ (r=1,2,3,\ldots)$$

$$(b)\ r(r-m)\gamma_{\eta+1,r} = 2\{p\gamma_{\eta,r} + q(\gamma_{\eta,r-1} + \gamma_{\eta,r+1})\},(r\leq\tfrac{m}{2})$$

$$(c)\ r(m+r)\alpha_{m+\eta+1,r} + \tfrac{1}{2}(m+2r)\delta_{\eta+1,r} = \{p\alpha_{m+\eta,r}$$

$$+q(\alpha_{m+\eta,r+1} + \alpha_{m+\eta,r-1}\},\begin{pmatrix} r=1,2.3,\ldots \\ \eta\geq 1 \end{pmatrix}$$

but when $n<m$,

$$r(m+r)\alpha_{n,r} = 2\{p\alpha_{n,r} + q(\alpha_{n,r+1} + \alpha_{n,r-1})\}.$$

$$(d)\ r(r-m)\beta_{m+\eta+1,r} + \tfrac{1}{2}(m-2r)\gamma_{\eta+1,r} = 2\{p\beta_{m+\eta,r}$$

$$+q(\beta_{m+\eta,r-1} + \beta_{m+\eta,\,r+1})\},(r\geq\tfrac{m}{2}),$$

but when $n<m$,

$$r(r-m)\beta_{n+1,r} = 2\{p\beta_{n,r} + q(\beta_{n,r-1} + \beta_{n,r+1})\},(r\geq\tfrac{m}{2}).$$

with the following restrictions :—

$$\alpha_{n,r} = \beta_{n,r} = \gamma_{n,r} = \delta_{n,r} = 0, \text{ if } r>n,$$

and also

$$\alpha_{n,o} = \beta_{n,o} = \gamma_{n,o} = 0, \text{ whatever } n \text{ is.}$$

§ 9

If we denote

$$\sum_{\eta=r}^{\infty}\delta_{\eta,r}=D_r;\quad \sum_{\eta=r}^{\infty}\gamma_{\eta,r}=C_r,$$

then D_r and C_r give the sum of the coefficients of $z\cos(m+2r)z$, and $z\cos(m-2r)z$ respectively in $\sum U_n(z)$.

Hence from (*a*) and (*b*) §8, we get

$$r(m+r)D_r=2\{pD_r+q(D_{r-1}+D_{r+1})\} \qquad \dots \quad (25)$$

$$r(r-m)C_r=2\{pC_r+q(C_{r-1}+C_{r+1})\} \qquad \dots \quad (26)$$

From the forms of (25) and (26) it is evident that the series for D_r and C_r are both convergent* and that

$$\lim_{r=\infty} D_r=0 \text{ and } \lim_{r=\infty} C_r=0.$$

Similarly, denoting

$$\sum_{n=r}^{\infty}\alpha_{n,r}=A_r\ ;\ \sum_{n=r}^{\infty}\beta_{n,r}=B_r,$$

which represent respectively the coefficients of $\sin(m+2r)z$ and $\sin(m-2r)z$, occuring in $\sum U_n(z)$, we get from (*c*) and (*d*)

$$r(m+r)A_r+\tfrac{1}{2}(m+2r)D_r=2\{pA_r+q(A_{r-1}+A_{r+1})\} \qquad \dots \quad (27)$$

$$r(r-m)B_r+\tfrac{1}{2}(m-2r)C_r=2\{pB_r+q(B_{r-1}+B_{r+1})\} \qquad \dots \quad (28)$$

Writing

$$w_r=-2q/\{r(m+r)-2p\},\quad w'_r=-2q/\{r(r-m)-2p\},$$

$$v_r=(m+2r)/2\{r(m+r)-2p\},\quad v'_r=(m-2r)/\{r(r-m)-2p\},$$

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* Whittaker and Watson · *Modern Analysis*, pp. 415-16.

we get from (27) and (28) respectively

$$v_r D_r + w_r A_{r-1} + A_r + w_r A_{r+1} = 0 \qquad \ldots \quad (29)$$

$$v'_r C_r + w'_r B_{r-1} + B_r + w'_r B_{r+1} = 0 \qquad \ldots \quad (30)$$

Eliminating $A_1, A_2, \ldots A_{r-1}, A_{r+1}, \ldots$ from (29), we get

$$A_r = -\frac{(-1)^r}{\Delta_0} \square_r, \qquad \ldots \quad (31)$$

where

$$\Delta_0 \equiv \left| \begin{array}{ccccccc} 1 & w_1 & & & & & \\ w_2 & 1 & w_2 & & & & \\ 0 & w_3 & 1 & w_3 & & & \\ 0 & 0 & w_4 & 1 & w_4 & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \right|$$

$$\square_r \equiv \left| \begin{array}{ccccccccccccc} v_1 D_1 + w_1, & 1 & w_1 & & & & & & & & & & \\ v_2 D_2, & w_2 & 1 & w_2 & & & & & & & & & \\ v_3 D_3, & 0 & w_3 & 1 & w_3 & & & & & & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & & & & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & & & & & \\ v_{r-1} D_{r-1}, & 0 & 0 & 0 & 0 & w_{r-1} & 1 & 0 & & & & & \\ v_r D_r, & 0 & 0 & 0 & 0 & 0 & w_r & w_r & 0 & & & & \\ v_{r+1} D_{r+1}, & 0 & 0 & 0 & 0 & 0 & 0 & 1 & w_{r+1} & 0 & & & \\ v_{r+2} D_{r+2}, & 0 & 0 & 0 & 0 & 0 & 0 & w_{r+2} & 1 & w_{r-2} & 0 & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \right|$$

since $A_0 = 1$.

Expanding the determinants $\square_r$ in terms of the elements of the first column, we get

$$A_r = \frac{(-1)^r}{\Delta_3}\left\{(v_1 D_2 + w_1)M_1 + \sum_{k=2}^{\infty} v_k D_k M_k \right\}. \qquad \dots \quad (32)$$

where the M's donote the first minors of the elements of the first column.

§ 10

It will be easily seen that with $s \geq 1$,

$$(i)\quad M_{r+s} = (-1)^{r+s-1} \cdot w_r w_{r+1} \dots w_{r+s-1} \cdot \Delta_{r+s} \begin{vmatrix} 1 & w_1 & & & & \\ w_2 & 1 & w_2 & & & \\ 0 & w_3 & 1 & w_2 & & \\ \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & w_{r-1} & 1 \end{vmatrix}$$

where Δ_{r+s} stands for the infinite determinant

$$\begin{vmatrix} 1 & w_{r+s+1} & & & & \\ w_{r+s+2} & 1 & w_{r+s+2} & & & \\ 0 & w_{r+s+3} & 1 & w_{r+s+3} & & \\ \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

and also when $k < r$,

$$(ii)\quad M_k = (-1)^{k-1} w_{k+1} \cdot w_{k+2} \dots w_r \cdot \Delta_r \begin{vmatrix} 1 & w_1 & & & & \\ w_2 & 1 & w_2 & & & \\ 0 & w_3 & 1 & w_3 & & \\ \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & w_{k-1} & 1 \end{vmatrix}$$

and further

$$(iii)\ \mathrm{M}_r=(-1)^{r-1}\left|\begin{array}{cccccccccc}
1 & w_1 & & & & & & & & \\
w_2 & 1 & w_2 & & & & & & & \\
0 & w_3 & 1 & w_3 & & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & & \\
\cdots & \cdots & \cdots & \cdots & w_{r-1} & 1 & 0 & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & w_{r+1} & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & w_{r+2} & 1 & w_{r+2} & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right|.$$

The infinite determinant Δ_r is convergent,* whatever r is and further $\lim_{r=\infty}.\Delta_r=1$. Hence the M's are all finite and the series (32) is therefore convergent and converges to a finite value, if r is finite; but vanishes if r is made infinitely large.

The same may be proved for B_r by means of the relation (30).

The series $\Sigma \mathrm{U}_n(z)$ is, clearly a series of analytical function of 'z' and can now easily be proved to be unformly convergent in any bounded domain of z and the process of term by term differentiation that we have carried out in §4 is valid.

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* Helge Von Koch: *Acta Mathematica*, Vol. XVI.

On an Extension of Sonine's Integral in Bessel Functions

BY

ABANIBHUSAN DATTA

[Read, January 6th, 1921]

Some very remarkable definite integrals involving Bessel functions have been evaluated by Sonine in an elaborate memoir published in *Mathematische Annalen*, Bd. 16. In particular he proved that

$$\int_0^\infty J_m(ax)J_m(bx)J_m(cx)\frac{dx}{x^{m-1}}$$

$$=0 \text{ or } \frac{[(a+b+c)(a+b-c)(b+c-a)(c+a-b)]^{m-\frac{1}{2}}}{\sqrt{\pi}2^{3m-1}\Pi(m-\frac{1}{2})a^m b^m c^m} \quad (m>-\tfrac{1}{2})$$

according as we can *not* or can form a triangle having the sides a, b, c. Sometimes in January, 1920, I undertook at the suggestion of Prof. Banerjee an investigation of similar integrals involving four or more Bessel functions of same order but different arguments. After the paper was written out and handed over to Prof Banerjee my attention was drawn by him to a paper on the same subject published by Prof. J. W. Nicholson[1] in a recent issue of the Quarterly Journal. Since then I have attempted to verify some of the results obtained by me with those of Nicholson and I find that in some cases his results differ substantially from those of mine. The method used by me is a simple extension of the one used by Sonine in proving the theorem quoted above whereas Prof. Nicholson has used a method suggested by Prof. Macdonald.[2] The difference in the results obtained by Nicholson and those of mine is due to certain errors which seem to have crept in some of the steps of his proof. I find as by Nicholson that the value of the integral

$$\int_0^\infty J_m(ax)J_m(bx)J_m(cx)J_m(dx)\frac{dx}{x^{2m-1}}$$

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[1] J. W. Nicholson "Generalisation of a Theorem due to Sonine," *Quar. Journ. of Mathematics*, Vol. 48, Part IV (1920), p. 321.

[2] Proc. Lond. Math. Soc., XXXV, 1902.

is zero when a, b, c, d do not form a quadrilateral. When they do form a quadrilateral the integral cannot be evaluated in simple forms but can be easily expressed in an infinite series.

2. Sonine has given, in his memoir mentioned above, the following elegant formula[1] :—

$$\int_0^\infty J_m(px)J_m(qx)e^{-hx^2}x\,dx$$

$$=\frac{p^mq^m}{\sqrt{\pi}2^m\Pi(m-\frac{1}{2})}\,\frac{1}{(2h)^{m+1}}\int_{-1}^{1}e^{-\frac{p^2+q^2-2pqt}{4h}}(1-t^2)^{m-\frac{1}{2}}dt$$

$$m>-\tfrac{1}{2}.$$

Substituting for h, $h+\frac{c}{2u}$, multiplying both sides by $\frac{1}{2\pi}e^{\frac{cu}{2}}\frac{dr}{u^{m+1}}$ and integrating between the limits $-\infty$ and $+\infty$, and putting $K=\frac{p^mq^n}{\sqrt{\pi}2^m\Pi(m-\frac{1}{2})}$, and $v^2=p^2+q^2-2pqt$, we have

$$\int_0^\infty J_m(px)J_m(qx)\,x\,e^{-hx^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\frac{cu}{2}-\frac{cx^2}{2u}}\,\frac{dr}{u^{m+1}}$$

$$=K\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\,dt\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\frac{cu}{2}-\frac{v^2}{4\left(h+\frac{c}{2u}\right)}}\frac{dr}{\left(2h+\frac{c}{u}\right)^{m+1}u^{m+1}},$$

where $u=k+ri$ and $m>-\frac{1}{2}$.

The left hand side can be easily proved to be equal to

$$\int_0^\infty J_m(px)J_m(qx)J_m(cx)e^{-hx^2}\frac{dx}{x^{m-1}},$$

by substituting cx for x in the formula[2]

$$J_n(x)=\left(\frac{x}{a}\right)^n\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\frac{a}{2}(k+ri)-\frac{x^2}{2a(k+ri)}}\frac{dr}{(k+ri)^{n+1}}. \quad \cdots \quad \text{(A)}$$

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[1] Sonine *loc cit*. Art. 44.

[2] Sonine *loc. cit*. Art. 25

where a is a real positive constant and n is greater than zero. The right hand side can be transformed into

$$K\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\,dt\,\frac{e^{-\frac{c^2+v^2}{4h}}}{2\pi\cdot(2h)^{m+1}}$$

$$\int_{-\infty}^{\infty}e^{\frac{c}{2}\left(u+\frac{c}{2h}\right)+\frac{cv^2}{8h^2\left(u+\frac{c}{2h}\right)}}\frac{dr}{\left(u+\frac{c}{2h}\right)^{m+1}}$$

As it is evidently justifiable to replace within the integral taken with respect to r, $u+\frac{c}{2h}$ simply by u, which is merely equivalent to changing of the constant k in $u=k+ri$, the second membe reduces to

$$K\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\,\frac{e^{-\frac{c^2+v^2}{2h}}}{2hv^m}\,i^m J_m\left(\frac{cv}{2hi}\right)dt.$$

Accordingly

$$\int_{0}^{\infty}J_m(px)J_m(qx)J_m(cx)e^{-hx^2}\frac{dx}{x^{m-1}}$$

$$=K\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\,\frac{e^{-\frac{c^2+v^2}{4h}}}{2hv^m}\,i^m J_m\left(\frac{cv}{2hi}\right)dt$$

$$=K\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\,\frac{e^{-\frac{c^2+v^2}{4h}}}{2hv^m}\cdot\frac{c^m v^m i^m\,dt}{(2h)^m i^m 2^m\sqrt{\pi}\,\Pi\,(m-\frac{1}{2})}$$

$$\int_{-1}^{1}e^{\frac{cv}{2h}z}(1-z^2)^{m-\frac{1}{2}}dz$$

(replacing $J_m\left(\frac{cv}{2hi}\right)$ on the right hand side by a definite integral)

$$=KL\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\,\frac{e^{-\frac{c^2+v^2}{4h}}}{(2h)^{m+1}}\,dt\int_{-1}^{1}e^{\frac{cvz}{2h}}(1-z^2)^{m-\frac{1}{2}}dz.$$

$$=\mathrm{KL}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\frac{1}{(2h)^{m+1}}$$

$$\int_{-1}^{1}e^{-\frac{c^2+v^2-2cvz}{4h}}(1-z^2)^{m-\frac{1}{2}}dz$$

where $$\mathrm{L}=\frac{c^m}{2^m\sqrt{\pi}\,\Pi\,(m-\frac{1}{2})}.$$

Substituting for h, $h=\dfrac{b}{2u}$, and then multiplying both sides by $\dfrac{1}{2\pi}e^{\frac{bu}{2}-\frac{by^2}{2u}}\dfrac{dr}{u^{n+1}}$ and integrating with regard to r between the limits $-\infty$ and $+\infty$, we have

$$\int_0^\infty \mathrm{J}_m(px)\,\mathrm{J}_m(qx)\,\mathrm{J}_m(cx)\,\frac{dx}{x^{m-1}}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\frac{bu}{2}-\frac{b(x^2+y^2)}{2u}}\frac{dr}{u^{n+1}}$$

$$=\mathrm{KL}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}dt\int_{-1}^{1}(1-z^2)^{m-\frac{1}{2}}dz$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\frac{bu}{2}-\frac{u(c^2+v^2-2cvz)}{2b}-\frac{by^2}{2u}}\times\left(\frac{u}{b}\right)^{m+1}\frac{dr}{u^{n+1}}$$

$$=\mathrm{KL}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}dt\int_{-1}^{1}(1-z^2)^{m-\frac{1}{2}}dz$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\frac{bu}{2}-\frac{u(c^2+v^2-2cvz)}{2b}-\frac{b^2y^2}{2bu}}\times\frac{1}{b^{m+1}}\frac{dr}{u^{n-m}}.$$

On performing the integration on both sides, we get by means of the formula (A)

$$\int_0^\infty \mathrm{J}_m(px)\,\mathrm{J}_m(qx)\,\mathrm{J}_m(cx)\,\frac{1}{x^{m-1}}\,\frac{\mathrm{J}_n(b\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2})^n}\,dx$$

$$=\mathrm{KL}b^{-n}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}dt\int_{a}^{1}(1-z^2)^{m-\frac{1}{2}}$$

$$\mathrm{J}_{n-m-1}\left(y\sqrt{b^2-c^2-v^2+2cvz}\right)\times\left(\sqrt{b^2-c^2-v^2+2cvz}\right)^{n-m-1}\cdot y^{m-n+1}\,dz;$$

$$n>m>-\tfrac{1}{2},$$

where

$$a=1 \text{ for } b^2<(c-v)^2,$$

$$a=\frac{c^2+v^2-b^2}{2cv} \text{ for } (c-v)^2<b^2<(c+v)^2,$$

$$a=-1 \text{ for } b^2>(c+v)^2.$$

For $y=0$, $n=m+1$, the above formula reduces to

$$\int_{0}^{\infty}\mathrm{J}_m(px)\,\mathrm{J}_m(qx)\,\mathrm{J}_m(cx)\,\mathrm{J}_{m+1}(bx)\,\frac{dx}{x^{2m}}$$

$$=\mathrm{KL}b^{-m-1}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}dt\int_{a}^{1}(1-z^2)^{m-\frac{1}{2}}dz.$$

Multiplying both sides of this formula by b^{m+1} and then differentiating with regard to b, we shall obtain

$$\int_{0}^{\infty}\mathrm{J}_m(px)\,\mathrm{J}_m(qx)\,\mathrm{J}_m(cx)\,\mathrm{J}_m(bx)\,\frac{dx}{x^{2m-1}}$$

$$=\mathrm{KL}b^{-m-1}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}(1-\beta_1{}^2)^{m-\frac{1}{2}}\frac{b}{cv}\,dt \text{ or zero} \quad \ldots \quad \text{(B)}$$

according as we can form a triangle having b, c, v for its sides, or not. In the above integral $\beta_1=\dfrac{c^2+v^2-b^2}{2cv}$.

These integrals can be evaluated by the method of quadratures. Otherwise we may proceed as follows:—

Expanding $\dfrac{1}{v}(1-\beta_1{}^2)^{m-\frac{1}{2}}$ by means of Taylor's Theorem in a series of t, and substituting the series thus obtained in the above integral, it becomes

$$\mathrm{KL}b^{-m-1}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}\sum_{n=0}^{n=\infty}\frac{t^n}{n!}\frac{d^n}{dt^n}\left[\frac{1}{v}(1-\beta_1{}^2)^{m-\frac{1}{2}}\right]_{t=0}dt$$

$$=\sum_{n=0}^{n=\infty}\mathrm{KL}b^{-m-1}\frac{d^n}{dt^n}\left[\frac{1}{v}(1-\beta_1{}^2)^{m-\frac{1}{2}}\right]_{t=0}.$$

$$\frac{1}{n!}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}t^n\cdot dt$$

$$=\sum_{n=0}^{n=\infty}-\mathrm{KL}\frac{b^{-m-1}}{n!}\frac{d^{2n}}{dt^{2n}}\left[\frac{1}{v}(1-\beta_1{}^2)^{m-\frac{1}{2}}\right]_{t=0}$$

$$\frac{(1\cdot3\cdot5\cdot\ldots(2n-1)\cdot1\cdot3\cdot5\cdot\ldots2m-1}{2\cdot4\cdot6\cdot\ldots(2m+2n)}\cdot\frac{\pi}{2}.$$

It is evident from the above that there are certain relations between p, q, c, b, t, and β_1. These can be better interpreted geometrically.

Since $\beta_1=\dfrac{c^2+v^2-b^2}{2cv}$ and $t=\dfrac{p^2+q^2-v^2}{2pq}$,

it is clear that β_1 is the cosine of the angle, of the triangle having c, v, and b for its sides, included between the sides c and v and similarly t is the cosine of the angle, of the triangle having p, q and v as its sides, included between p and q. Hence it is obvious from the adjoining figure that p, q, c, and b form a quadrilateral having v as one of its diagonals, t and β_1 being the cosines of the angles θ and ϕ respectively.

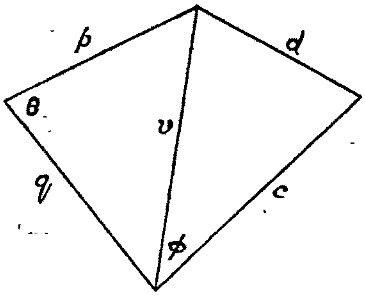

Fig. 1.

The integral on the right hand side is equal to $\mathrm{KL}b^{-m-1}$

$$\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}(1-\beta_1{}^2)^{m-\frac{1}{2}}\frac{b}{cv}\,dt \text{ or zero}$$

(1) according as we can from a triangle having b, c and v as its sides or not $\left(\because \beta^1=\dfrac{c^2+v^2-b^2}{2cv}\right)$,

and also (2) according as we can from a triangle having p, q and v for its sides $\left(\because t=\frac{p^2+q^2-v^2}{2pq}\right)$, because v must always be a real positive quantity,

and hence according as we can form a quadrilateral having p, q, c and b for its sides, v being the diagonal common to the two triangles thus formed.

With the help of the geometrical interpretation given above, the formula (B) can be transformed thus :—

$$\text{Since } t=\cos\theta, \qquad dt=-\sin\theta\, d\theta$$

$$\beta_1=\cos\phi, \qquad v^2=p^2+q^2-2pqt$$

$$\int_0^\infty J_m(px)\, J_m(qx)\, J_m(cx)\, J_m(bx)\, \frac{dx}{x^{2m-1}}$$

$$=\frac{p^m q^n c^m b^{-m-1}}{\pi 2^{2m}\{\Pi(m-\frac{1}{2})\}^2}\int_{-1}^{1}(1-t^2)^{m-\frac{1}{2}}(1-\beta_1{}^2)^{m-\frac{1}{2}}\frac{b}{cv}\,dt$$

$$=-\frac{\mathrm{KL}b^{-m}}{c}\int_{+\pi}^{0}(1-\cos^2\theta)^{m-\frac{1}{2}}(1-\cos^2\phi)^{m-\frac{1}{2}}\frac{\sin\theta d\theta}{(p^2+q^2-2pq\cos\theta)^{\frac{1}{2}}},$$

$$=-\mathrm{KL}\,\frac{b^{-m}}{c}\int_{+\pi}^{0}(\sin\theta)^{2m}(\sin\phi)^{2m-1}\frac{d\theta}{(p^2+q^2-2pq\cos\theta)^{\frac{1}{2}}}$$

$$\text{or} \quad -\mathrm{AB}\,\frac{q^{-m}}{p}\int_{+\pi}^{0}(\sin\gamma)^{2m}(\sin\delta)^{2m-1}\frac{d\gamma}{(c^2+d^2-2cd\sin\gamma)^{\frac{1}{2}}}$$

for $q\sin\theta\sin\phi=b\sin\gamma\sin\delta$, A, B being corresponding functions.

From the general integral with different arguments and involving Bessel Functions of order $m>-\frac{1}{2}$, we can deduce some interesting results by imposing particular relationship between the arguments.

Suppose, for example, $p=q=c=b$, in which case, the quadrilateral becomes a rhombus and we have

$$\theta=\gamma, \quad \delta=\phi.$$

Putting $\frac{\theta}{2}=\psi$, we have

$$\phi+\psi=90^\circ. \quad \theta=2\psi,$$

$$\sin\phi=\cos\psi,$$

$$\sin\theta=2\sin\psi\cos\psi.$$

$$d\theta=2d\psi.$$

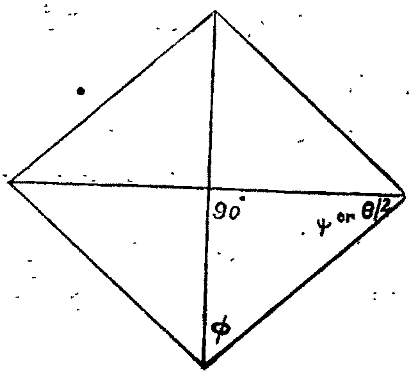

Fig. 2.

$$(p^2+q^2-2pq\cos\theta)^{\frac{1}{2}}=\sqrt{2}p(1-\cos\theta)^{\frac{1}{2}}=\sqrt{2}p\,(2\sin^2\psi)^{\frac{1}{2}}=2p\sin\psi.$$

Hence $$\int_0^\infty \frac{\{J_m(px)\}^4\,dx}{x^{2m-1}}=\frac{KLb^{-m}}{c}\int_{\frac{\pi}{2}}^{0} -2^{2m}\sin^{2m}\psi\cos^{2m}\psi\cos^{2m-1}\psi\,\frac{d\psi}{p\sin\psi}$$

$$=-\frac{KLb^{-m}}{c}2^{2m}\int_{\frac{\pi}{2}}^{0}(\cos\psi)^{4m-1}(\sin\psi)^{2m-1}\,dt$$

$$=\frac{a^{2m-2}}{\pi\{\Pi(m-\frac{1}{2})\}^2}\int_0^{\frac{\pi}{2}}(\cos\psi)^{4m-1}(\sin\psi)^{2m-1}\,d\psi$$

$$=\frac{a^{2m-2}}{2\pi\{\Pi(m-\frac{1}{2})\}^2}\cdot\frac{\Gamma(2m)\Gamma(m)}{\Gamma(3m)}.$$

Form the particular case, in which $p=q$, and $c=b$, we have

$$\int_0^\infty J_m{}^2(px)\,J_m{}^2(cx)\,\frac{dx}{x^{2m-1}}=\frac{p^{2m-1}c^{-1}}{\pi\{\Pi(m-\frac{1}{2})\}^2}$$

$$\int_0^{\frac{\pi}{2}}\sin^{2m-1}\psi\cos^{2m}\psi\left(1-\frac{p^2}{c^2}\sin^2\psi\right)^{m-\frac{1}{2}}d\psi$$

Nicholson however finds that

$$\int_0^\infty J_m^4(ax)\,\frac{dx}{x^{2m-1}}=\frac{a^{2m-2}}{2\pi}\,\frac{1}{\Gamma^2(m+\frac{1}{2})}\int_0^{\frac{\pi}{2}}\cos^{4m-1}\theta\,\sin^{m-1}\theta\,d\theta$$

$$=\frac{a^{2m-2}}{4\pi}\,\frac{\Gamma(2m)\,\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{5m}{2}\right)\Gamma^2(m+\frac{1}{2})}$$

which seems to me to be incorrect. The generalisation, that if $m>-\frac{1}{2}$, then

$$\int_0^\infty J_m(a_1x)\,J_m(a_2x)\ldots J_m(a_nx)\,x^{-mn+2m+1}\,dx=0$$

provided that $(a_1, a_2,\ldots a_n)$ do not form the sides of a polygon, is easily obtained.

It seems to me that the discrepancy between the results obtained by Nicholson and that of mine is to be attributed to his incorrectly quoting the formula

$$\frac{J_m^2(ax)}{x^m}=\frac{a^m}{\sqrt{\pi}2^{m-1}\omega(m-\frac{1}{2})}\int_0^{\frac{\pi}{2}}\cos^{2m}\theta\,J_m(2\,ax\sin\theta)\,d\theta.$$

The correct formula can be easily obtained from the following result due to Gegenbauer

$$J_m(pc)\,J_m(qc)=\frac{p^mq^mc^m}{\sqrt{\pi}2^m\Pi(m-\frac{1}{2})}\int_{-1}^{1}\frac{J_m(c\sqrt{p^2+q^2-2pqt})}{(\sqrt{p^2+q^2-2pqt})^m}(1-t^2)^{m-\frac{1}{2}}dt$$

$m>-\frac{1}{2}$.

Putting $p=q=a,\quad c=x,\quad t=\cos\theta.$

$$J_m^2(ax)=-\frac{a^{2m}x^m}{\sqrt{\pi}\,2^m\Pi(m-\frac{1}{2})}\int_\pi^0\frac{J_m(x\sqrt{2a^2-2a^2\cos\theta})}{(2a^2-2a^2\cos\theta)^{\frac{m}{2}}}(1-\cos^2\theta)^{m-\frac{1}{2}}\sin\theta\,d\theta$$

$$=-\frac{a^{2m}x^m}{\sqrt{\pi}2^m\Pi(m-\frac{1}{2})}\int_\pi^0\frac{J_m\left(2ax\sin\frac{\theta}{2}\right)}{\left(2a\sin\frac{\theta}{2}\right)^m}\sin^{2m}\theta\,d\theta.$$

Putting $\frac{\theta}{2}=\psi$, we get $J_m^2\,(ax)$

$$=-\frac{a^{2m}x^m}{\sqrt{\pi}2^m\Pi(m-\frac{1}{2})}\int_{\frac{\pi}{2}}^{0}\frac{J_m\,(2ax\sin\psi)}{(2a\sin\psi)^m}\quad(2\sin\psi\cos\psi)^{2m}\,2d\psi$$

$$=-\frac{a^{2m}x^m\,2^{2m}.\,2}{\sqrt{\pi}2^m\Pi(m-\frac{1}{2})a^m}\int_{\frac{\pi}{2}}^{0}J_m\,(ax\sin\psi)\sin^m\psi\cos^{2m}\psi\,d\psi.$$

$$=\frac{2a^m\,x^m}{\sqrt{\pi}\Pi(m-\frac{1}{2})}\int_{0}^{\frac{\pi}{2}}J_m\,(2ax\sin\psi)\sin^m\psi\cos^{2m}\psi\,d\psi.$$

Using this correct formula, the value of the integral

$$\int_{0}^{\infty}\frac{J_m^4(ax)}{x^{2m-1}}\,dx$$

is given by

$$2\,\frac{Aa^m}{\sqrt{\pi}\Pi(m-\frac{1}{2})}\int_{0}^{\frac{\pi}{2}}\cos^{2m}\psi\,(2a\sin\psi)^{m-1}\sin^m\psi\,(4a^2-4a^2\sin^2\psi)^{m-\frac{1}{2}}\,d\psi$$

where $A=\sqrt{\pi}2^{3m-1}\,a^{2m}\Pi(m-\frac{1}{2})$.

or $$\frac{a^{2m-2}}{\pi\{\Pi(m-\frac{1}{2})\}^2}\int_{0}^{\frac{\pi}{2}}\cos^{4m-1}\psi\sin^{2m-1}\psi\,dt$$

or $$\frac{a^{2m-2}}{\pi\{\Pi(m-\frac{1}{2})\}^2}\;\frac{\Gamma(2m)\Gamma(m)}{2\Gamma(3m)}$$

which agrees with the result obtained by me.

My best thanks are due to Prof. S. K. Banerji for his suggesting this *problem* to me and also for the interest he has taken in my work.

On Some Symmetric Determinants*

BY

HARIPADA DATTA.

[*Read, January 16th, 1921.*]

I. It is well-known that all centro—symmetric determinants can be expressed as the product of two other determinants.

To effect this resolution into factors, we are to perform on the determinants the operations :

(1) $\text{row}_1+\text{row}_n$; $\text{row}_2+\text{row}_{n-1}$; $\text{row}_3+\text{row}_{n-2}$; ...

and then on the resulting determinant the operations,

(2) $\text{col}_n-\text{col}_1$; $\text{col}_{n-1}-\text{col}_2$; $\text{col}_{n-2}-\text{col}_3$; ...

Here it is to be shown that there are determinants, besides the centro-symmetric ones, which can be expressed, by the same operations, as products of two other determinants.

With the operations given above, we obtain

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\ g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+g_1 & a_2+g_2 & a_3+g_3 & a_4+g_4-a_3-g_3 & a_4+g_5-a_2-g_2 & a_6+g_6-a_1-g_1 \\ b_1+f_1 & b_2+f_2 & b_3+f_3 & b_4+f_4-b_3-f_3 & b_5+f_5-b_2-f_2 & b_6+f_6-b_1-f_1 \\ c_1+d_1 & c_2+d_2 & c_3+d_3 & c_4+d_4-c_3-d_3 & c_5+d_5-c_2-d_2 & c_6+d_6-c_1-d_1 \\ d_1 & d_2 & d_3 & d_4-d_3 & d_5-d_2 & d_6-d_1 \\ f_1 & f_2 & f_3 & f_4-f_3 & f_5-f_2 & f_6-f_1 \\ g_1 & g_2 & g_3 & g_4-g_3 & g_5-g_2 & g_6-g_1 \end{vmatrix}$$

---

* This paper was written on 1st November, 1917, when I was working in Cambridge, as a Carniegi Research Student.

This determinant

$$=\begin{vmatrix} a_1+g_1 & a_2+g_2 & a_3+g_3 \\ b_1+f_1 & b_2+f_2 & b_3+f_3 \\ c_1+d_1 & c_2+d_2 & c_3+d_3 \end{vmatrix} \times \begin{vmatrix} d_4-d_3 & d_5-d_2 & d_6-d_1 \\ f_4-f_3 & f_6-f_2 & f_6-f_1 \\ g_1-g_3 & g_5-g_2 & g_6-g_1 \end{vmatrix}$$

provided that

$$a_4+g_4-a_3-g_3=a_5+g_5-a_2-g_2=a_6+g_6-a_1-g_1$$
$$=b_4+f_4-b_3-f_3=b_5+f_5-b_2-f_2=b_6+f_6-b_1-f_1$$
$$=c_4+d_4-c_3-d_2=c_5+d_5-c_2-d_2=c_6+d_6-c_1-d_1=0.$$

These nine relations can be satisfied in more than one way, for everyone of them is quite independent of the other eight.

Then these relations are satisfied in the following way :—

$$\left.\begin{matrix} a_1=g_6\ ;\ a_6=g_1 \\ b_1=f_6\ ;\ b_6=f_1 \\ c_1=d_6\ ;\ c_6=d_1 \end{matrix}\right\}\ \left.\begin{matrix} a_2=g_5 \quad a_5=g_2 \\ b_2=f_5\ ;\ b_5=f_2 \\ c_1=d_5\ ;\ d_5=c_2 \end{matrix}\right\}\ \left.\begin{matrix} a_3=g_4\ ;\ a_4=g_3 \\ b_3=f_4\ ;\ b_4=f_3 \\ c_3=d_4\ ;\ c_4=d_3 \end{matrix}\right\}$$

The determinant is centro-symmetric.

The following determinant which is not centro-symmetric, is resolvable into three factors each of which is a determinant :—

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 \\ -d_1 & d_7 & -d_3 & d_5 & d_4 & -d_6 & d_2 & -d_8 \\ c_8 & -c_2 & c_6 & -c_4 & -c_5 & c_2 & -c_7 & c_1 \\ -b_1 & b_7 & -b_3 & b_5 & b_4 & -b_6 & b_2 & -b_8 \\ a_8 & -a_2 & a_6 & -a_4 & -a_5 & a_3 & -a_7 & a_1 \end{vmatrix}$$

$$=\begin{vmatrix} a_1+a_1 & a_3+a_6 \\ c_1+c_8 & c_5+c_6 \end{vmatrix} \times \begin{vmatrix} b_2+b_7 & b_4+b_5 \\ d_2+d_7 & d_4+d \end{vmatrix} \times \begin{vmatrix} d_4-d_5 & d_3-d_6 & d_2-d_7 & d_1-d_8 \\ c_1-c_5 & c_3-c_6 & c_2-c_7 & c_1-c_8 \\ b_4-b_6 & b_4-b_1 & b_2-b_7 & b_1-b_8 \\ a_4-a_5 & a_3-a_6 & a_2-a_7 & a_1-a_5 \end{vmatrix}$$

By interchanging the place of a_1 with a_8, and of a_3 with a_6; and altering the signs of the rest of the elements of the first row we obtain the last row. By interchanging the place of b_2 with b_7, and of b_4 with b_5, altering the signs of the rest of the elements of the second row, we obtain the last but one row. Carrying on this procsss of interchanging of places and altering in signs alternately the other rows are obtained. For half the determinant, the elements, from bottom upwards, are alternately positive and negative, for the elements which change places retain their signs. Of this determinant, only half of the elements occupy centro-symmetric positions while the other half do not (here order of the determinant being $4n$).

(*i*) Even if the signs of those elements (a_2, a_4, b_1 etc.) are not altered. still the determinant is resolvable into three factors.

(*ii*) If

$$\begin{aligned} &d_1,\ c_1=a_4,\ b_4; &&d_2,\ c_2=a_3:\ b_3\\ &d_3,\ c_3=a_2,\ b_2 &&d_4,\ c_4=a_1.\ b_2\\ &d_5,\ c_5=a_8,\ b_8; &&d_6:\ c_6=a_7,\ c_7\\ &d_7,\ c_7=a_6,\ b_6; &&d_8,\ c_8=a_5,\ b_6 \end{aligned}$$

Then the third factor becomes centro-symmetric and therefore breaks up into two factors. Hence the determinant—

$$\begin{vmatrix} a_1+a_8 & a_3+a_6 \\ b_4+b_5 & b_2+b_7 \end{vmatrix} \times \begin{vmatrix} a_3+a_6 & a_1+a_8 \\ b_2+b_7 & b_4+b_5 \end{vmatrix}$$

$$\times \begin{vmatrix} a_2-a_3+a_6-a_7 & a_1-a_4+a_5-a_8 \\ b_2-b_3+b_6-b_7 & b_1-b_4+b_5-b_8 \end{vmatrix} + \begin{vmatrix} a_1+a_4-a_5-a_8 & a_2+a_3-a_6-a_7 \\ b_1+b_4-b_5-b_8 & b_2+b_3-b_6-b_7 \end{vmatrix}$$

The first two factors differ in sign only.

(*iii*) If the last row is obtained from the first in the same way as the last but one from the second, and the last but three from the third in the same way as the last but three from the fourth, and so on (*i.e.*, if the proçess of interchanging of places and altering in signs, is same throughout except the $(2n+2)$th row then the determinant of order $4n+2$ vanishes while the determinant of order $4n$, is still resolvable into three factors.

(*iv*) Exactly in the similar way the determinant of odd order, may be treated.

2. Let D_6*

$$=\begin{vmatrix} a & a+1 & a+2 & a+3 & a+4 & a+5 \\ a+1 & a+2 & a+3 & a+4 & a+5 & a+6 \\ a+2 & a+3 & a+4 & a+5 & a+4 & a+3 \\ a+3 & a+4 & a+5 & a+4 & a+3 & a+2 \\ a+4 & a+5 & a+4 & a+3 & a+2 & a+1 \\ a+5 & a+4 & a+3 & a+2 & a+1 & a \end{vmatrix}$$

$$=\begin{vmatrix} 2a+5 & 2a+5 & 2a+5 \\ 2a+5 & 2a+7 & 2a+7 \\ 2a+5 & 2a+7 & 2a+9 \end{vmatrix} \times \begin{vmatrix} -1 & -1 & -1 \\ -1 & -3 & -3 \\ -1 & -3 & -5 \end{vmatrix}$$

$$=\begin{vmatrix} 2a+5 & 0 & 0 \\ 2a+5 & 2 & 0 \\ 2a+5 & 2 & 2 \end{vmatrix} \times \begin{vmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ -1 & -2 & -2 \end{vmatrix} =(-1)^3 2^6 (2a+5).$$

Similarly

$$D_{2n}=(-1)^n 2^{2(n-1)}(2a+2n-1).$$

3. Let K_{n+1}

$$=\begin{vmatrix} a_{1\,n+1} & a_{1\,n} & a_{1\,2} & a_{1\,1} \\ a_{2\,n+1} & a_{2\,n} & a_{2\,2} & a_{2\,1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+1,\,n+1} & a_{n+1,\,n} & a_{n+1,\,2} & a_{n+1,\,2} \end{vmatrix}$$

and ${}^rK_n$ the determinant obtained from it by deleting the $(r+1)th$ column and the last row. And also let $b_n^{(r)} = {}^rK_n/K_n$.

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* *See "On Symmetric Determinants and Paffians, Art. 6. Proc Edin. Math. Soc. (Part 2) 1915-16.*

The recurrent

$$\begin{vmatrix} b_1^{(1)} & 1 & & & & \\ b_2^{(2)} & b_2^{(1)} & 1 & & & \\ b_3^{(3)} & b_3^{(2)} & b_3^{(1)} & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ b_{n+1}^{(n-1)} & b_{n-1}^{(n-2)} & \cdots & \cdots & b_{n-1}^{(1)} & 1 \\ b_n^{(n)} & b_n^{(n-1)} & \cdots & \cdots & b_n^{(2)} & b_n^{(1)} \end{vmatrix} = a_{1\,n+1}.$$

For if on this determinant the operation

$$a_{11}\text{col}_1 - a_{12}\text{col}_2 + a_{13}\text{col}_3 - \ldots + (-1)^{n-1}\text{col}_{1n}$$

be performed then all the elements, except the last one, of the first column, vanish, and this last element is $(-1)^{n+1}a_{1\,n+1}$.

Therefore, if $a_{1\,n+1}=0$ then the whole reecurrent will vanish.

(*i*) If $b_r^{(m)} = \dfrac{m(m-1)\ldots(m-r+1)}{1\cdot 2\ldots r}$

$$\times\ \frac{(b+m)(b+m-1)\ \ldots\ (b+m-r+1)}{(a+2m-1)(a+2m-2)\ldots(a+2m-r)}$$

then the value of the recurrent is

$$\frac{(b+1)(b+2)\ldots(b+n)}{(a+1)(a+2)\ldots(a+n)}.$$

Here the operation is

$$\text{col}_1 \frac{b+1}{a+1}\text{col}_2 + \frac{(b+1)(b+2)}{(a+1)(a+2)}\text{col}_2 - \ldots$$

$$+ (-1)^{n-1}\frac{(b+1)\ldots(b+n-1)}{(a+1)\ldots(a+n-1)}\text{col}_n.$$

4. If c_n

$$
=\begin{vmatrix}
a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\
a_2 & a_1 & 1 & 0 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_{n-1} & \dots & \dots & \dots & \dots & a_1 & 1 \\
a_n & \dots & \dots & \dots & \dots & a_2 & a_1
\end{vmatrix}
$$

then the persymmetric determinant of order n

$$
\begin{vmatrix}
c_2 & c_3 & \dots & c_{n+1} \\
c_3 & c_4 & \dots & c_{n+2} \\
\dots & \dots & \dots & \dots \\
c_{n+1} & \dots & \dots & c_{2n}
\end{vmatrix}
=(-1)^n
\begin{vmatrix}
1 & a_1 & a_2 & \dots & a_n \\
a_1 & a_2 & a_3 & \dots & a_{n+1} \\
\dots & \dots & . & . & \dots \\
a_n & a_{n+1} & \dots & \dots & a_{2n}
\end{vmatrix}
$$

To prove this let us take a determinant of oder 4, namely

$$
\begin{vmatrix}
c_2 & c_3 & c_n & c_4 \\
c_3 & c_4 & c_5 & c_6 \\
c_4 & c_5 & c_6 & c_7 \\
c_5 & c_6 & c_7 & c_8
\end{vmatrix}
$$

If we multiply this determinant by

$$
\begin{vmatrix}
1 & 0 & 0 & 0 \\
-a_1 & 1 & 0 & 0 \\
a_2 & -a_1 & 1 & 0 \\
-a_3 & a_2 & -a_1 & 1
\end{vmatrix}
\qquad \dots \quad (1)
$$

and the resulting determinant by the same determinant (1) (in the first place the multiplication is to be performed row by row and in the second place column by row), then we obtain

$$
\begin{vmatrix}
a_1^2-a_2 & a_1a_2-a_3 & a_1a_3-a_4 & a_1a_4-a_5 \\
a_1a_2-a_3 & a_2^2-a_4 & a_2a_3-a_5 & a_2a_4-a_6 \\
a_1a_3-a_4 & a_2a_3-a_5 & a_3^2-a_6 & a_3a_4-a_7 \\
a_1a_4-a_5 & a_2a_4-a_6 & a_3a_4-a_7 & a_4^2-a_8
\end{vmatrix}
\qquad \dots \quad (2)
$$

an axis-symmetric determinant. By the addition of a column and a row (2) can be written as

$$\left|\begin{array}{c|cccc} 1 & 0 & 0 \cdots 0 & & 0 \\ \hline a_1 & & & & \\ a_2 & \multicolumn{4}{c}{\text{the determinant (2)}} \\ a_3 & & & & \\ a_4 & & & & \end{array}\right| \qquad \ldots (3)$$

Performing on this determinant (3) the operations

$$\text{col}_2 - a_1\text{col}_1\,;\ \text{col}_3 - a_2\text{col}_1\,;\ \text{col}_4 - a_3\text{col}_1\,;\ \text{col}_5 - a_4\text{col}_1$$

we obtain

$$(-1)\begin{vmatrix} 1 & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ a_4 & a_5 & a_6 & a_7 & a_8 \end{vmatrix}.$$

The order of this determinant is less, by unity, than the sum of the suffix of first c of the first row and the order of the original determinant in c's. As for example

$$\begin{vmatrix} c_4 & c_5 & c_6 & c_7 \\ c_5 & c_6 & c_7 & c_8 \\ c_6 & c_7 & c_8 & c_9 \\ c_7 & c_8 & c_9 & c_{10} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & a_1 & a_2 & a_2 & a_4 \\ a_1 & 1 & 0 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_1 & 1 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_2 & a_1 & a_4 & a_5 & a_6 & a_7 \\ a_4 & a_3 & a_2 & a_5 & a_6 & a_7 & a_8 \\ a_5 & a_5 & a_3 & a_6 & a_7 & a_8 & a_9 \\ a_6 & a_5 & a_4 & a_7 & a_8 & a_9 & a_{13} \end{vmatrix}$$

That the last determinant is persymmertic becomes evident if we interchange the first column with the third. In general

$$\begin{vmatrix} c_r & c_{r+n} & \cdots & c_{r+n-1} \\ c_{r+1} & c_{r+2} & \cdots & c_{r+n} \\ \cdots & \cdots & \cdots & \cdot \\ c_{r+n-1} & \cdots & \cdots & c_{r+2n-2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & a_1 & \cdots & a_n \\ 0 & 0 & \cdots & 0 & a_1 & a_2 & \cdots & a_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_1 & \cdots & \cdots & \cdots & \cdots & \cdots & a_{n+r-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & a_{n+1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{2n+r-2} \end{vmatrix} (n+r-1)$$

multiplied by a sign factor $(-1)^{\frac{1}{2}(r-1)}$ or $(-1)^{\frac{1}{2}r}$ according as r odd or even.

Seismograms and their Significance[1]

BY

SUDHANSUKUMAR BANERJI

1. Introduction. In opening a discussion on the interpretation of seismograms, I am pained to observe that although India has witnessed in recent times some of the most devastating earthquakes, such as the Great Indian Earthquake[2] of 1897 or the Kangra Earthquake of 1905 which wrought considerable mischief especially in the epicentral area and resulted in the loss of more than 20,000 lives, Indians do not appear to have evinced much interest into the scientific study of earthquake phenomena. So far as I am aware the only valuable works on earthquakes done in this country were by two Europeans, namely, R. D. Oldham[3] and C. S. Middlemiss both of whom were Superintendents of the Geological Survey of India. Now that the Indians are forging their heads in scientific research, I should consider myself fortunate if the discussion which is being opened to-day will excite in any of them an interest into this recently developed branch of science pregnant with immense possibilities.

In two papers[4] published in the Philosophical Magazine for July, 1916, and January, 1918, I investigated the properties of sound waves produced by the impact of two bodies, say two billiard balls. The sound waves so produced are highly impulsive in character and similar is the case with a great tectonic earthquake produced by a relief of the strain to which the rock masses of the earth are subjected by mountain making or other crustal movements. But the analogy drawn between a disturbance travelling through a homogeneous gaseous medium and one travelling through a complicated structure like that of the earth must necessarily be far-fetched. Attempts were made fully described in the papers quoted above to obtain records of the character of the impulsive sound waves by various methods and although results were obtained confirming the broad indications of theory, the photographs obtained could not be regarded as satisfactory records of the character of the sound pulse owing to the free vibrations of the receiving apparatus excited by the sound of the impact. Arrangements to damp the free vibrations of the receiving instrument proved ineffective and rendered the instrument less sensitive. Very similar has been the experience of those pioneers in the study of seismological science who constructed the various types of seismographs with a view to record the earth movement. To avoid the free vibrations of the pendulum rods used in these instruments, devices have been employed to make the rods practically aperiodic or dead beat. Although aperiodicity is obtained at the expense of sensitiveness, the great energy contained in the earthquake waves has not rendered the seismographs in which damping arrangements are used altogether insensitive. We are therefore perhaps safe in assuming that our seismographs especially the Galitzin types in which electromagnetic dampers are used give reliable records for the horizontal and vertical components of the ground movement within the limits of error introduced by instrumental complications. It should, however, be noted that even with the best damping devices such for example as we have in the Galitzin arrangements, the free vibrations of the pendulum rod are not altogether avoided. This will be clear from the fact that two instruments of the same type will seldom give identical records of the same earthquake even if installed in the same room. All seismograms must therefore be considered liable to error to a certain extent on this account.

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[1] Opening address in a discussion of 'Seismograms and their Significance' delivered at the Eighth *Indian Science Congress* held at Calcutta on February 1st, 1921. The results of an enquiry by the present author into the mathematical theories of earthquake waves read before the Calcutta Mathematical Society on 4th April, 1920, have been incorporated in this address.

[2] The epicentral tract for this earthquake was situated in the neighbouring province of Assam.

[3] R. D. Oldham, "Report on the Great Earthquake of 12th June, 1897," *Mem. Geol. Surv. India*, XXIX, 1899; "On the propagation of earthquake to great distances," *Phil. Trans.*, A., Vol 194 (1900), p. 135.

[4] S Banerji, "Aerial Waves Generated by Impact," Part I, *Phil. Mag.*, July, 1916 and Part II, *Phil Mag*, January, 1918.

2. The three principal types of Wave-Motion in an Isotropic Elastic Solid.

It was in 1831 that Poisson discovered that in an isotropic elastic solid there are two types of waves travelling with different velocities. That was a great discovery in the theory of earthquake waves although the importance of it was hardly recognised at the time when it was made. The recognition of the irrotational and the equivoluminal character of these two types of waves is due to Stokes. It is well known that the equations of motion of an isotropic elastic solid can be written in the form

$$(\lambda+2\mu)\nabla^2\Delta=\rho\frac{\partial^2\Delta}{\partial t^2},$$

$$\mu\nabla^2(\omega_x, \omega_y, \omega_z)=\rho\frac{\partial^2}{\partial t^2}(\omega_x, \omega_y, \omega_z),$$

where Δ denotes the dilatation and ω_x, ω_y, ω_z the components of rotation, λ and μ being elastic constants for the medium The waves of dilatation or longitudinal waves therefore travel through the medium with a velocity $\left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}}$ and the waves of distortion or transversal waves involving rotation without dilatation travel with velocity $\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$. A very important pronouncement in the theory of the subject under discussion was made by Lord Rayleigh in the year 1887 when he showed that besides the above two types of waves there can travel on the surface of an elastic solid a third type of waves with a velocity which is slightly less than the waves of distortion. Taking $z=0$ to be coincident with the solid Lord Rayleigh showed that for progressive waves whose fronts are parallel to the axis of y, the displacements, if the material be incompressible can be expressed in the form

$$u=U(e^{-fz}-\cdot 5433\,e^{-sz})\sin(pt+fx),$$

$$w=U(e^{-fz}-1\cdot 840\,e^{-sz})\cos(pt+fx),$$

where U is a constant and the velocity of propagation is

$$\frac{p}{f}=\cdot 9554\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$$

and if the meterial satisfies Poisson's conditions

$$u=U(e^{-rz}-\cdot 5773\,e^{-sz})\sin(pt+fx)$$

$$w=U(\cdot 1475\,e^{-rz}-1\cdot 4679\,e^{-sz})\cos(pt+fx)$$

and the velocity of propagation is

$$\cdot 9194\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}.$$

The disturbance represented by these expressions is confined to a superficial region comparable with the wave length and as they diverge in two dimensions only they must gain an increasing relative importance at a considesable distance from the source.

3. Division of the Seismograms into three phases.

A seismogram or instrumental time-record of of the earth movement produced by an earthquake at some distance is now easily divided into three main parts [1], known as the 'first phase' the 'second phase' and the 'long wave phase' respectively. The beginning of the first phase is denoted by P and the beginning of the second phase by S and these are interpreted as representing the arrival of longitudinal and transversal waves which start simultaneously at the focus of the earthquake and travel by *brachistochronic* paths with different velocities and reach the observing station at different times The beginning of the long wave phase is denoted by L and is identified with the arrival of the surface waves with a velocity agreeing closely with that of Rayleigh waves. If the earthquake is a very large one we may after about 2½ hours observe the arrival of long waves that have travelled by the opposite sides of the earth.

The fact that P and S represent the arrival of longitudinal and transversal waves that have travelled by brachistochronic paths from focus to the observing station is easily understood. It should not however be supposed that the waves of these two types travel by two simple brachistochronic paths from focus to station. This would undoubtedly be the case had the earth been a homogeneous isotropic elastic solid but the multiple diffraction which these waves must undergo in the heterogeneous mass of rocks that constitute the earth's crust must give rise not only to one principal but many subsidiary brachistochronic paths from focus to the station. This will partly explain the irregular movements that follow P and S. These features may also be partly due to subsidiary shocks either at the primary focus or at other points.

The long wave phase though found by measurement to travel round the earth's surface with a velocity agreeing closely with that of Rayleigh waves is a complex phenomenon and "For distances not less than two thousand kilometres the general appearance of this phase is marked by first a few waves of *period about 20 seconds,* gradually increasing in amplitude and looking as if they had been drawn with a shaking hand, then a rapid development of extremely smoth waves of rather shorter period which reach a maximum amplitude, subside, pass through a succession of maxima before merging into the tail of the earthquake or coda. For short distances however this description does not hold good, L succeeds S very quickly, *shorter periods of about 12* o face page 243.

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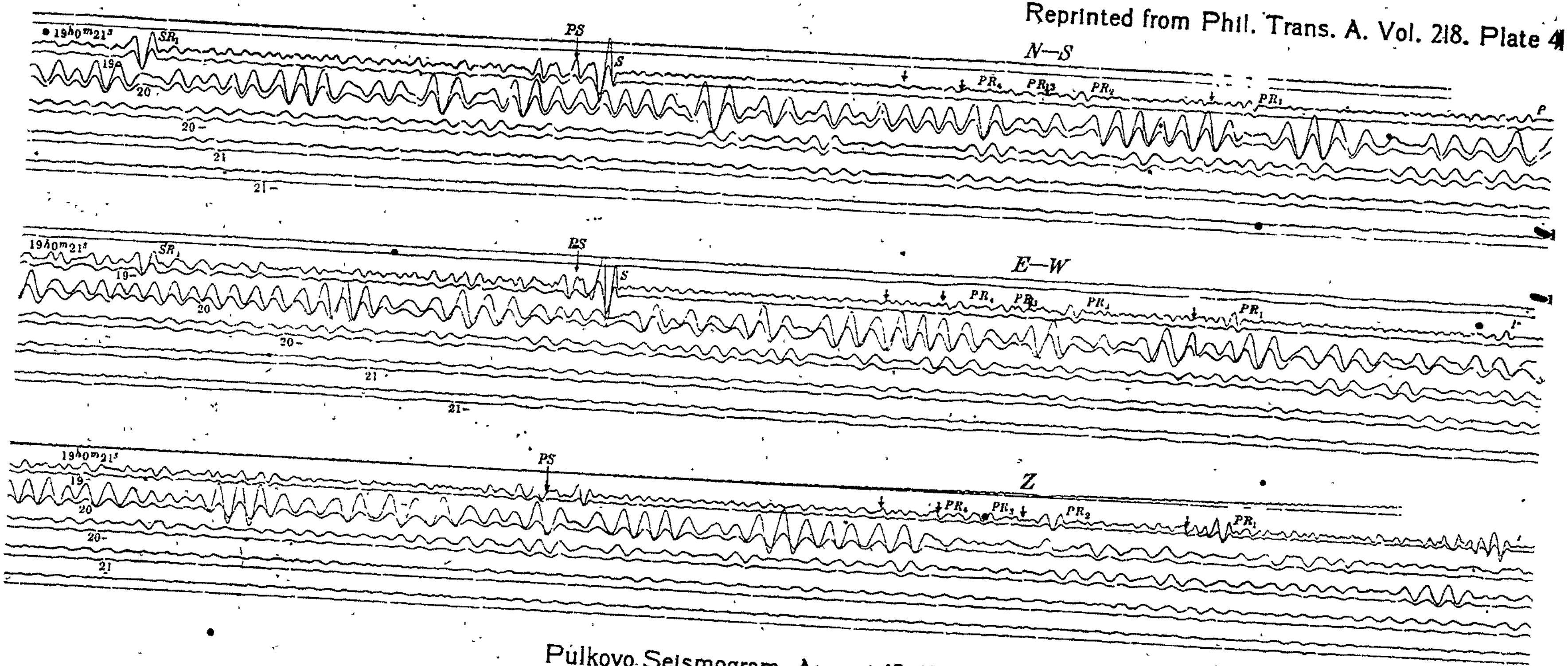
[1] Oldham "On the propagation of earthquake to great distances," *Phil. Trans.* A. Vol. 194 (1900) p. 135. See also G. W. Walker "Modern Seismology," p. 48.

Reprinted from Phil. Trans. A. Vol. 218. Plate 4



Púlkovo Seismogram, August 15, 1913.
Deduced Epicentre, Δ = 8540 km., φ = 27° N., λ = 142° E.

seconds prevail, and the duration of the whole phase becomes very short. Following the maximal or long wave phase we have the Coda. The amplitudes are row small and the movements are somewhat irregular and lacking in smoothness. But *still the motion here is on the whole periodic and sinusoidal (about 12 seconds).*" [1] The remarkable periodicity of the long wave phase in the seismogram and of the coda is rather difficult to understand in view of the fact that the primary disturbance is an *impulse* and has led to some conjectures about the surface crust of the earth. The most interesting of these hypotheses is that of Wiechert [2] who supposed that the surface crust of the earth assumed to have a thickness of half the wave length of surface waves or about 35 km.,and to be resting on a sheet of plastic material is set into free vibrations by the impulse of the earthquake waves. Such a crust by its natural vibrations will provide an explanation of the dominant period of the long wave phase, say 20 seconds. But objection has been raised to the hypothesis of a plastic material on astronomical grounds. Further if such a crust is supposed to exist at all it must not only confine within itself the surface waves but also the longitudinal and the transversal waves. But we have definite reasons to assume that these two latter types of waves penetrate deep into the earth. And is the hypothesis of a plastic meterial an absolutely essential one ? The surface crust of the earth exposed to daily radiations into the outside space may eventually be conceived to be in a different physical state from that of the interior subject to an enormous pressure due to the surface crust. The familiar analogy of the bark and the trunk within is naturally brought into mind which need not necessarily be plastic and may be perfectly elastic.

4. Surface reflexions of earthquake waves.[3]

We have seen how the notation P and S is used to denote the arrival of longitudinal and transversal waves as recorded in the seismograms, by brachistochronic paths. But a seismogram may also show well marked sharp impulses comprised within the first and the second phases. Wiechert supposed that these represent the arrival of waves that have undergone one or more reflexions at the earth's surface. The notation PR_n is used to denote a longitudinal wave that has been reflected n times at the surface thus dividing the epicentral distance into

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[1] G. W. Walker " Modern Seismology," p. 50.

[2] Wiechert and Zoppritz, " Ueber Erdbeben Wellen," *Gott. Nach.*, 1907.

[3] Walker, " Surface Reflexions of Earthquake waves " *Phil Trans.*, Ser A, Vol. 218 (1919). p. 373.

$(n+1)$ equal parts and a similar notation SR$_n$ is used for the transversal waves. Waves are also contemplated of the type P$_n$ S$_m$ called *Wechselwellen* by Wiechert that have undergone change from longitudinal to transversal *or vice versa* on reflexion at the earth's surface.

Recently Walker in a paper published in the Philosophical Transactions of the Royal Society for 1919 has undertaken a quantitative analysis of reflexions theory on simple lines to explain some of the difficulties that have hitherto attended the interpretation of seismograms and suggests that a systematic examination of both times and magnitudes of P and S and at least PR$_1$, SR$_1$ and PS or SP is likely to afford important information about the primary time curves. The difficulties mainly examined in this paper are : (1) that the times of the arrival of the reflected wave systems are not always in agreement with theoretical calculation from the curves for P and S as functions of epicentral distances obtained by Zöppritz[1] which are in general use at seismological observatories, (2) that the relative magnitudes of the reflected waves to P and S undergo changes at different distances that are difficult to understand, for example, it is doubtful if reflexions are shown at all at less than 3000 km., at 8000 km. PR$_1$ is about as large as P, while at 12000 km. PR$_1$ is substantially larger than P.

It is not difficult to understand why no reflexions should occur for epicentral distances less than 3000 km. For such distances the seismic rays will be practically confined within the surface crust of the earth. For greater distances it seems to me that the simple reflexion theory is insufficient to explain the phenomena. It would appear from the manner in which the relative magnitudes of the reflected waves change at different distances, that the absorption of seismic rays within the earth play a far more important part than it has hitherto been supposed to do. If it is assumed that the absorption increases with the depth then it is not difficult to conceive that a seismic ray which has followed a direct route from the focus to station will be more absorbed having entered deep into the earth than the one which has been reflected one or more times at the earth's surface and therefore pursues a course more or less close to the surface curst.

Assuming the loss of motion by communication to be proportional to the velocity, the elastic equations of motion can be written in the form

$$\frac{\lambda+\mu}{\rho}\,\frac{\partial\Delta}{\partial x}+\frac{\mu}{\rho}\nabla^2 u=\ddot{u}+k\dot{u}$$

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[1] Zöppritz, *Gott. Nach.*, 1907.

and two similar equations for v and w. These equations reduce to

$$\frac{\lambda+2\mu}{\rho}\nabla^2\Delta=\ddot{\Delta}+k\dot{\Delta}$$

and
$$\frac{\mu}{\rho}\nabla^2\omega_x=\ddot{\omega}_x+k\dot{\omega}_x$$

and two similar equations for ω_y, ω_z. The solutions of these equations can be written in the form

$$\Delta=\Sigma_n A_n\frac{\sin nr}{r}e^{-\frac{1}{2}kt}\cos\left\{\sqrt{\left(\frac{\lambda+2\mu}{\rho}n^2-\tfrac{1}{4}k^2\right)}\,t-a\right\}$$

$$\omega_x=\Sigma_n B_n\frac{\sin nr}{r}e^{-\frac{1}{2}kt}\cos\left\{\sqrt{\left(\frac{\mu}{\rho}n^2-\tfrac{1}{4}k^2\right)}\,t-a\right\},$$

k may be supposed to slowly vary with the depth and the factor $e^{-\frac{1}{2}kt}$ will determine the decay of vibrations.

Another point to which attention should be invited at this stage is that although the possibility of a longitudinal or transversal wave changing into a longitudinal or transversal wave or *vice versa* on reflexion has been recognised by Walker in his recent investigation, there seems to be however no reason why a longitudinal or a transversal wave should not give rise on reflexion to both of these types of waves simultaneously, nay, to Rayleigh waves as well.

5. Energy carried away by the three types of waves.

It is now definitely recognised that ' all earthquakes are due to the sudden fracture of the rock of the earth's crust which has been strained by slow earth movements beyond its strength.[1] Strong vibrations are set up at the fractured surface at the time of the fracture ' and these travel as longitudinal and transversal waves through the elastic solid material of the earth having a velocity of 7 and 4 kms. near the earth's surface. This is very well understood but how are the Rayleigh waves which travel over the surface of the earth originated? The Rayleigh waves must be started when the body waves mentioned above arrive at the surface. An idea of how and when they are formed can be obtained from the following considerations :—

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[1] See the section on "the problems of seismology" by H. F. Reid under "A Survey of Research Problems in Geophysics" prepared by the American Geophysical Union, *Proc. of the National Academy of Sciences*, Vol. 6, October (1920), p. 556.

Let the crust by whose fracture the earthquake waves are generated occupy a region of space which we denote by T. Suppose that this region had initially no displacement and that a velocity ($\dot{u}_0, \dot{v}_0, \dot{w}_0$) was suddenly given throughout T by the fracture of the crust. It is further supposed that at this instant ($\dot{u}_0, \dot{v}_0, \dot{w}_0$) has zero values outside T. Let V_1, V_2, donote the velocities of propagation of longitudinal and transversal waves. Let S_1 denote a sphere of radius V_1t having its centre at any point (x, y, z) outside T and S_2 a sphere of rodius V_2t having its centre at the same point. Let Ω denote that part of the volume containod between these two spheres which is within T. Let r denote the distance of any point (x', y', z') within Ω or on the parts of S_1 and S_2 that are within T from the point (x, y, z) and let $\dot{q}_0$ denote the sudden velocity acquired at the point (x', y', z') projected upon the radius vector r, supposed drawn from x, y, z. Then the displacement (u, v, w) at the point (x, y, z) at the instant t can be written in the form[1]

$$u=\frac{1}{4\pi}\iiint t\left\{\dot{u}_0\frac{\partial^2 r^{-1}}{\partial x^2}+\dot{v}_0\frac{\partial^2 r^{-1}}{\partial x\partial y}+\dot{w}_0\frac{\partial^2 r^{-1}}{\partial x\partial z}\right\}d\Omega$$

$$+\frac{1}{4\pi}\iint t\dot{q}_0\frac{\partial r^{-1}}{\partial x}dS_1-\frac{1}{4\pi}\iint\left\{t\dot{q}_0\frac{\partial r^{-1}}{\partial x}-\frac{1}{r^2}t\dot{u}_0\right\}dS_2$$

and similar expressions for v and w can be easily written down.

Obviously the disturbance expressed by (u, v, w) will first reach the part on the earth's surface which we call epicentral tract and the disturbance on this tract will influence the formation of a set of surface waves travelling with the usual velocity of Rayleigh Waves. Professor Knott has shown that the greater part of the energy of an earthquake comes to the earth's surface within a short distance of the epicentre and 95 % within a distance of 90°. It would thus appear that a considerable part of the energy that arrives at the earth's surface within the epicentral tract or in its neighbourhood is converted into surface waves. 'When we remember that the earthquake originates at a fault and that the mass movement there may be horizontal, inclined, or vertical, or of the nature of a twist, we realize that this movement will influence one type of waves more than the others and that the energy sent out in different azimuths and at different vertical angles must be dependant on the direction of the fault and of the nature of the mass movement.'[1]

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[1] See Love's *Elasticity*, Third Edition, Art. 211, p. 308 (1920). Reference may also be made to a paper by Love in *London Math. Soc. Proceedings* (Ser. 2), Vol. I (1904), p. 291.

If the focus of the earthquake be taken as the origin and if it is supposed that the earthquake is due to a single force of magnitude $\chi(t)$ acting in a direction which we take to be the axis of x, then the displacement (u_o, v_o, w_o) at any point before the disturbance has reached the curved surface of the earth is given by

$$u_o = \frac{1}{4\pi\rho}\,\frac{\partial^2 r^{-1}}{\partial x^2}\int_{r/a}^{r/b} t'\chi(t-t')\,dt' + \frac{1}{4\pi\rho r}\left(\frac{\partial r}{\partial x}\right)^2\left\{\frac{1}{a^2}\chi\left(t-\frac{r}{a}\right)\right.$$

$$\left.-\frac{1}{b^2}\,\chi\left(t-\frac{r}{b}\right)\right\}+\frac{1}{4\pi\rho b^2 r}\,\chi\left(t-\frac{r}{b}\right),$$

$$v_o = \frac{1}{4\pi\rho}\,\frac{\partial^2 r^{-1}}{\partial x\,\partial y}\int_{r/a}^{r/b} t'\chi(t-t')\,dt' + \frac{1}{4\pi\rho r}\frac{\partial r}{\partial x}\,\frac{\partial r}{\partial y}\left\{\frac{1}{a^3}\,\chi\left(t-\frac{r}{a}\right)\right.$$

$$\left.-\frac{1}{b^2}\,\chi\left(t-\frac{r}{b}\right)\right\},$$

$$w_o = \frac{1}{4\pi\rho}\,\frac{\partial^2 r^{-1}}{\partial x\,\partial z}\int_{r/a}^{r/b} t'\chi(t-t')\,dt + \frac{\partial r}{\partial x}\,\frac{\partial r}{\partial z}\left\{\frac{1}{a^2}\,\chi\left(t-\frac{r}{a}\right)\right.$$

$$\left.-\frac{1}{b^2}\,\chi\left(t-\frac{r}{b}\right)\right\},$$

a and b denoting the velocities of propagation of longitudinal and transversal waves.[2]

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[1] Proc. Nat. Acd. of Sciences, *loc. cit.*

[2] Love's *Elasticity*, (Third Edition), p. 310.

If the origin be transferred from the focus to the centre of the earth and the axes be taken to be a set of parallel axes drawn at that point, the same expressions for (u_0, v_0, w_0) remain true provided we change the meaning of r to

$$r^2=(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2,$$

α, β, γ now denoting the coordinates of the earthquake focus.

When the disturbance represented by (u_0, v_0, w_0) reaches the earth's surface, it will not satisfy the condition of zero stress there. To satisfy this condition of zero stress, we assume that there exists simultaneously a set of disturbance given by

$$u_0=\Sigma_n\left(A_n\frac{\partial}{\partial x}+B_n\frac{\partial}{\partial y}+C_n\frac{\partial}{\partial z}\right)^n\chi\left(t-\frac{R}{a}\right)$$

$$+\Sigma_n\left(F_n\frac{\partial}{\partial x}+G_n\frac{\partial}{\partial y}+H_n\frac{\partial}{\partial z}\right)^n\chi\left(t-\frac{R}{b}\right)$$

and two similar expressions for v_1 and w_1, where R is the radius vector from the earth's centre, and a set of surface waves of the Rayleigh types given by

$$u_2=\Sigma_n\left(A_n'\frac{\partial}{\partial x}+B_n'\frac{\partial}{\partial y}+C_n'\frac{\partial}{\partial z}\right)^n e^{-k\left(t-\frac{R-\rho}{c}\right)}$$

$$\chi\left(t-\frac{R-\rho}{c}\right)$$

and two similar expressions for v_2, w_2, c being the velocity of propagation of Rayleigh waves and ρ the earth's radius.

Expanding (u_0, v_0, w_0) by Taylor's theorem in the form

$$u_0=\Sigma_n\ L_n\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}+\gamma\frac{\partial}{\partial z}\right)^n\chi\left(t-\frac{R}{a}\right)$$

$$+\Sigma_n M_n\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}+\gamma\frac{\partial}{\partial z}\right)^n\chi\left(t-\frac{R}{b}\right).$$

where L_n, M_n are known constants, the arbitrary constants are determined from the conditions of zero stress on the boundary, $R=\rho$, namely, three equations of the type

$$\frac{\lambda}{\mu} x \Delta + \frac{\partial}{\partial x}(ux+vy+wz)+R\frac{\partial u}{\partial R}-u=0$$

where $$u=u_0+u_1+u_2, \text{ etc.}$$

The detailed discussion of this problem will be given by the author in a subsequent paper.

It should however be remarked here that a very important investigation has been made by Professor Lamb[1] on the propagation of tremors over the surface of a semi-infinite elastic solid bounded by a plane. The following expressions have been obtained by him for the horizontal and vertical displacements due to a single impulse of the type

$$R(t)=\frac{\overline{R}}{\pi}\,\frac{\tau}{t^2+\tau^2}$$

at a distance large compared with $\frac{\tau}{c}$:

$$q_o=\frac{H}{\pi\mu}\frac{\partial}{\partial\omega}\int_0^\infty R\,(t-c\omega\cosh u)\,du-\frac{2}{\pi^2 b\mu\omega}\int_a^b \theta.\,U\,(\theta).$$

$$\frac{\partial}{\partial\theta}\int_0^\infty R\,(t-\theta\omega\cosh u)\,du\,d\theta,$$

$$w_o=\frac{P}{\pi^2 b\mu}\int_0^\infty \theta.V(\theta)\;\frac{\partial}{\partial t}\int_0^\infty R\,(t-\theta\omega\cosh u)\,du\,d\theta,$$

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[1] *Phil. Trans. Roy Soc.* (Ser. A), Vol. 203 (1904).

where
$$U(\theta)=-\frac{b^3\theta(2\theta^2-b^2)\sqrt{(\theta^2-a^2)}\sqrt{(b^2-\theta^2)}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)},$$

$$V(\theta)=\frac{b^3(2\theta^2-b^2)^2\sqrt{(\theta^2-a^2)}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)},$$

$\omega=\sqrt{(x^2+y^2)}$ and $a<\theta<b$.

Curves for q_0 and w_0 have been obtained from the asymptotic values of these integrals. It appears that "the initial form of the system of waves depends on the history of the primary impulse, but if this be of limited duration, the system gradually develops a characteristic form, marked by three salient features travelling with the velocities proper to irrotational, equivoluminal and Rayleigh waves respectively. As the wave system passes any point of the surface, the horizontal displacement shows first of all a single well marked oscillation followed by a period of comparative quiescence, and then another oscillation, corresponding to the epoch of arrival of equivoluminal waves. The whole of this stage constitutes what may be called 'the minor tremor;' it is of course more and more protracted the greater the distance from the source, and its amplitude continually diminishes, not only absolutely but also relatively to that of 'the main shock' which is identified with the arrival of the Rayleigh waves. Similar statement applies to the vertical displacement, except that the minor tremor leads up more gradually to the main shock, being interrupted, however, by a short of jirk at the epoch of arrival of equivoluminal waves." Prof. Lamb made an estimate of the effect of the curvature of the earth and pointed out that the chief qualitative difference introduced by the curvature of the earth will be that the minor tremor will be propagated directly through the earth, so that the first two epochs will (at distances comparable with the radius) be accelerated relatively to the main shock. It seems to me that Prof. Lamb very much underestimated the effect of the curvature of the earth. The curvature is mainly instrumental to produce the multiple reflexions of the earthquake waves recognised in the seismograms by that notations PR_n, SR_n and $P_n\,S_n$ and we have seen that some of these reflected disturbances are more important than the original disturbance itself.

6. Oceanic disturbance produced by submarine earthquakes.

It is well-known that submarine earthquakes give rise to great water waves. In the case of the Pacific Ocean it is known that these waves travel from one side of the ocean to the other. It was at one time the general belief that the first indication of these waves along a coast is marked by the withdrawal of water, followed by a great elevated wave. It is now recognised that this is not a general characteristic of the oceanic disturbance and although this order is very frequent, the elevated wave is very often seen first to appear.

It would be very interesting to give a mathematical discussion of the oceanic disturbance produced by earthquakes. But the problem is a complex one on account of the fact that the water waves and the surface earthquake waves travel with different velocities. If the disturbance in the epicentral tract under the sea (which by far will be more important than the subsequent surface tremors progressing over the bottom of the sea) be regarded as a source varying according to some prescribed law, say for example,

$$f(t) = \frac{\tau}{t^2 + \tau^2}$$

a law for which the surface tremors on a semi-infinite solid have been investigated by Lamb, then the tidal elevation ζ is given by[1]

$$2\pi g\zeta = \frac{1}{4\sqrt{2}\tau^2}\left(\frac{c\tau}{r}\right)^{\frac{1}{2}} \sin\left(\frac{1}{4}\pi - \frac{3}{2}\eta\right) \cos^{\frac{3}{2}}\eta$$

approximately, where $t = \frac{r}{c} + \tau \tan \eta$. The diagram giving the relation between ζ and t given in Lamb's *Hydrodynamics* (page 293) shows clearly how the elevation and the subsequent withdrawal of water as the wave progresses over any point are sharply defined. Whether the withdrawal should be preceded by elevation or *vice versa* will obviously depend on the position of the epicentre as well as the observed point.

7. Interior of the Earth.

Very valuable information has been obtained about the interior of the earth from the observed times of transit of seismic rays to different places on the earth's surface. The speeds of propagation of longitudinal and transversal waves depend on the elastic constants and the density and it has been suggested that this may provide us with a means of estimating the elastic constants of the material of which the earth is composed. The problem to be solved is the inverse one of calculating the speed of propagation at different points inside the earth from the observed times of transit to different places on the earth's surface. Rudzki and Kovesligethy treated this problem mathematically by regarding the rays along which the energy is propagated as brachistochronic paths, their mathematical assumptions, however, were made so as to get a soluble case, and had no dynamical harmony with known seismological facts. In 1907. Prof. C. G. Knott[1] discovered a law of velocity which gave results agreeing closely with observations, while Wiechert and Zöppritz worked out an interesting case in which it was assumed that the rays are circles.

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[1] Lamb's *Hydrodynamics* (Fourth Edition), Art. 197.

If it is assumed that the earth may be divided into a series of concentric spherical shells within each of which the material is uniform, it has been shown by Bateman[2] in 1910 that the problem reduces to an integral equation of the Abel type and that its solution can be easily obtained. If Δ denote the epicentral distance and T the time taken by the seismic ray to travel to the point, we have

$$\frac{\Delta}{R}=\theta=f(c)=2c\int_{c}^{b}(\eta^2-c^2)^{-\frac{1}{2}}\frac{d}{d\eta}\log r\,d\eta,$$

$$T=2\int_{c}^{b}(\eta^2-c^2)^{-\frac{1}{2}}\eta^2\frac{d}{d\eta}\log r\,d\eta$$

where $b=\frac{R}{v_o}$, v_o being the velocity of propagation at the surface, and

$\frac{p}{v}=c=\frac{R\cos e}{v_o}=R\frac{dT}{d\Delta}=\frac{dT}{d\theta}$, $\frac{r}{v}=\eta$, v being the velocity at a point r and R_o the earth's radius. The solution is given by

$$\log r=-\frac{1}{\pi}\int_{\eta}^{b}(c^2-\eta^2)^{-\frac{1}{2}}\,\theta\,dc$$

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[1] Knott, "The propagation of earthquake waves through the earth and connected problem," *Proc. Edin. Roy. Soc.*, Vol. 39, p. 157.

[2] Bateman, *Phil. Mag.*, 1910.

and $$\eta^2 \frac{d}{d\eta} \log r = -\frac{1}{\pi} \frac{d}{d\eta} \int_{\eta}^{b} (c^2 - \eta^2)^{-\frac{1}{2}} c \top dc,$$

so that if θ and $\top$ can be expressed as a function of c or $\frac{d\top}{d\theta}$ we should get r as a function of η and hence the velocity at any depth. Now the observations give $\top$ as a function of Δ, so that theoretically the problem is completely solved.

Following up this solution and comparing it with observed data, Prof. Knott only recently in 1919 in the *Proceedings of the Edinburgh Royal Society* have obtained very valuable conclusions about the interior of the earth. His results may be summarised as follows :—

(1) The seismic rays, both of the condensational and distortional waves, are on the whole concave outwards, indicating that the speeds of propagation increase with the depth below the earth's surface until a depth equal to about three-tenths of the earth's radius is reached.

(2) At this depth the speeds of propagation tend to or reach a constant value and then fall off slightly for greater depths, certain rays showing a convexity outwards.

(4) That data of observation are insufficient to enable us to trace waves which reach a depth lower than six-tenths of the earth's radius.

(4) The evidence is that at or near this depth the distortional wave is cut out, so that over arcual distances from the epicentre greater than 120° there is no characteristic appearance of the secondary wave in the seismograms.

(5) The hypothesis suggested by these facts and deductions is that the earth consists of an elastic solid shell down to a depth of about half the earth's radius, that at this depth, the rigidity begins to break down, and that finally at a depth of six-tenths of the earth's radius, the elastic solid shell gives place to a non-rigid nucleus of measurable compressiblity. This hypothesis is broadly similar to the views advanced by R. D. Oldham so early as 1901.

Records of earthquakes at distances more than 110° have proved very unsatisfactory. Doubts are still entertained whether this is due to the imperfections of our observations or to the existence of a central core in the earth having different physical properties from the material surrounding it as suggested in the above hypothesis.

8. Microseisms.

I do not propose to enter into a discussion of the various theories that have been advanced to explain microseisms or "earth tremors" which are still controversial. I would only refer to Dr. Klotz of Ottawa who studied the microseisms by means of a Bosch seismograph and is of opinion that the largest effects observed at Ottawa are always associated with the cyclonic areas in the North Atlantic and he suggests that the microseismic movement may appear in West Europe before the cyclone arrives. If this is so, it is a very valuable addition to our meteorological knowledge. If those who are responsible for our meteorological observatories most of which are provided with seismographs had learned to observe this effect many valuable lives and properties could perhaps have been saved in the terrible cyclone which visited East Bengal only the other day and whose memory is still fresh in our mind.

9. Earthquake Periodicity.

It is said that the Tibetans can predict the occurrence of earthquakes. If this be true the Tibetans must have discovered some kind of periodicity of the earthquake activity from earthquake statistics. It would be interesting to know what kind of periodicity the Tibetans have discovered. It may however be remarked that by the application of Fourier analysis to earthquake statistics Prof. Turner[1] already finds evidence of a 452 day period of earthquake activity.

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[1] See *Brit. Ass. Reports*, 1912.

Notes and News

At the Annual Convocation of the Calcutta University, held on March 24th, Lord Ronaldshay, Rector of the University, invited the Hon'ble Sir Asutosh Mookerjee to accept the Vice-Chancellorship of the University and to carry out the much needed reforms recommended by the Calcutta University Commission. Perhaps no other educationist in India is better fitted for this gigantic task.

The Calcutta University has nominated Prof. C. V. Raman, M.A., as the representative of their scientific teaching staff to attend the forthcoming Congress of the Universities of the British Empire which meets at Oxford in July next under the Presidency of Lord Curzon. Prof. Raman intends to sail for England in May and will visit the principal centres of scientific research in the British Isles. He will carry with him the heartiest good wishes of all scientific workers in India.

The death of Sir Rashbehary Ghose, the prince among the Calcutta Graduates, the foremost Lawyer of Bengal and the greatest benefactor of the Calcutta University, has cast a gloom over the whole of Bengal and rightly so, for any country in the world can be proud of such a mighty product of her soil. The Calcutta Mathematical Society have reasons to be especially grateful to the memory of Sir Rashbehary Ghose. At the time when the Society was just formed, a donation from him enabled the Society to build up its library.

The third Conference of the Indian Mathematical Society was held in Lahore on Saturday, Sunday and Monday, March 26th, 27th and 28th, 1921. His Excellency Sir Edward Douglas Maclagan, Governor of the Punjab opened the Conference. Twenty papers were communicated besides the Presidential Address by Mr. Balak Ram, M.A., I.C.S., and the Report of the Society by the Hon. Secretary, Prof. D. D. Kapadia, M.A., B.Sc.

The following five mathematical periodicals have ceased publication (temporarily, *at least*): *Archiv der Mathematik und Physik* (last volume, 28, 1919); *Bibliotheca Mathematica* (last volume, 14, 1914-15); *Journal de Mathématics Pures et Appliqueés* or *Liouville's Journal* (last volume 84, 1919); *Nouvelles Annales de Mathématiques* (last volume 79, 1920); *Zeitschrift für Mathematik und Physik* (last volume 64, 1917). It has been announced that *Annales scientifique de l' Ecole Normale Supérieure* and *Bulletin de la Société Mathématique de France* will have to be discontinued unless many new subscribers are forthcoming at an early date. The *London Mathematical Society* had recently to increase the membership fees and the admission fees from one guinea to two guineas with a view to meet the enormous deficit with which the Society had been running its journal. The post-war conditions will thus appear to have seriously affected some of the well-known mathematical journals.

CALCUTTA MATHEMATICAL SOCIETY

REPORT FOR THE YEAR 1920

1. The gentlemen named below were elected officers and other members of the Council for the year 1920 :—

President :

The Hon'ble Justice Sir Asutosh Mookerjee, Swaraswati, Kt., C.S.I., M.A., D.L., D.Sc., Ph.D., F.R.S.E.

Vice-Presidents :

Mr. Mahendranath Ray, C.I.E., M.A., B.L.
Dr. C. E. Cullis, M.A., Ph.D.
Dr. Syamadas Mookerjee, M.A., Ph.D.

Treasurer :

Rai Bahadur Abinaschandra Bose, M.A.

Secretary :

Dr. Sudhansukumar Banerjee, D.Sc.

Council :

Dr. D. N. Mallik, D.Sc., F.R.S.E.
Dr. S. C. Bagchi, LL.D., Bar-at-Law.
C. V. Raman, Esq , M.A.
Satischandra Bose, Esq., M.A.
Manmathanath Ray, Esq., M A., B.L.
Narendrakumar Majumdar, Esq., M.A.
Phanindralal Gangooly, Esq., M.A., B.L.
Surendramohan Gangooly, Esq., M.Sc.
Bibhutibhusan Datta, Esq., M.Sc.

2. During the year 1920, five meetings were held and 32 papers were communicated.

3. Four issues of the Bulletin (namely, Vol. X, No. 4; Vol. XI, Nos. 1, 2, 3) containing 24 original papers, reviews, notices and miscellaneous notes have deen published in 1920. The number of the Bulletin which was to have been published in June, 1920, could not

be brought out in time on account of illness of the Secretary and consequently a double number of the Bulletin was issued in September 1920. The four numbers of the Bulletin issued during the year form a volume exceeding 250 pages.

4. The most important event during the year was the holding of an International Congress of Mathematicians at Strasbourg on September 22, 1920. The Society was duly invited to participate in the function but it was not found possible to send any representative to Strasbourg. The Congress appears to have been a very successful function although on account of the peculiar situation in which the last war has left the world some countries found it impossible to send delegates to the Congress. Though attended chiefly by French Mathematicians, it appears that six Englishmen and twelve Americans took part in the programme. Italians, Spaniards and Scandinavians, as well as Germans and Austrians were conspicuously absent. The total attendance was about 200. It is to be hoped that the cloud which the last war has cast over the world will soon disappear and that the International Congress of 1924 which has been voted to be held in the United States of America will be a greater success and will bring together the mathematicians of the world to an everlasting bond of friendship and mutual good will.

5. During the year India has lost a distinguished mathematician by the death of Mr. Ramanujan, B.A., F.R.S. Mr. Ramanujan died at the early age of thirty-two and was the first Indian to become a Fellow of the Royal Society of London.

6. The Council adopted this year the important rule that a paper accepted for publication in the Bulletin of the Society should not be republished by the author in a different Journal without the sanction of the Council and without making the necessary acknowledgment. The Council regrets to find that this rule has not been usually observed by the contributors.

7. During the year the Society has received letters from several distinguished savants of Europe and America expressing their appreciation of the work done by the Society and extending their co-operation and fellowship for which the Society offers its best thanks. The Society has entered into several new exchange relations during the year. The Society's exchange relationship with the German Academies which was temporarily stopped on account of the war has been revived during this year. The present exchange list of the Society contains

nearly sixty different names. The Bulletin is being subscribed by some of the Universities of Europe and America.

8. The Bulletins of the Society have been regularly reviewed in the Revue Semestrielle des Publications Mathematiques, the American Mathematical Monthly, the Mathematical Gazette, the Science Abstracts and the Science Progress. The appointment of the present Secretary of the Society as a collaborator for the first mentioned publication has greatly helped in the regular review of the Bulletin by that publication. Prof. Dr. L. Lichtenstein has informed the Society that the forthcoming publication of the Fortschritte der Mathematik will contain a review of all papers published in the Bulletins of the Society. The Society has been recently requested for co-operation in the working organisation of the Zentrale für naturwissenschaftliche Berichterstattung, Berlin and the Society has thankfully accepted the offer.

9. As in previous years, the Society has to record its gratefulness to its distinguished founder President who has always given to the Society ungrudgingly and cheerfully a portion of his most precious time to the great encouragement of the members of the Society and whose inspiring presence was seldom missed in any of the meetings of the Society. The present mathematical activity in the University of Calcutta is his creation and it is only to be hoped that under his fostering care this activity amongst the Calcutta mathematicians will not only increase the yearly output of mathematical research in Bengal which will tend to enrich the Society's Bulletin but will also secure for it a recognised place amongst the best mathematical productions of the world. The Society also tenders its best thanks to the other officers and the members of the Council for the valuable services rendered by them to the Society.

10. The authorities of the Calcutta University Press recently communicated to the Society that they will henceforth charge the Society for the publication of its Bulletin only one-third the actual cost of its publication. For this act of patronage, the Society is deeply grateful to the authorities of the Press. The best thanks of the Society are also due to the authorities of the Press for expediting the printing of the Bulletin which has enabled the Society to issue four numbers during the year under review.

11. Ten ordinary members were elected in 1920. The present number of ordinary members of the Society is 170 and of honorary members 25.

APPENDIX A.

The following papers were read before the Calcutta Mathematical Society during the year 1920 :—

1. Prof. Sudhansukumar Banerji, D.Sc. "On the application of Mathieu functions to the determination of the vibrations of a string with a simple harmonic distribution of density."

2. Mr. Girindralal Mookerjee, M.A., B.L. "On the geometrical construction of a trinodal quartic."

3. Mr. Nripendranath Ghosh, M.A. "The potent divisors of the characteristic matrix of a minimum simple square ante-slope." [*Communicated by Prof. C. E. Cullis.*]

4. Mr. Bholanath Pal, M.A. "A note on Whittaker's formula for the solution of algebraic or transcendental equations."

5. Mr. Abanibhushan Datta, M.A. "On a generalisation of Neumann's expansion in a series of Bessel functions."

6. Mr. Nripendranath Sen, M.Sc. "Liquid motion inside rotating curvilinear rectangles."

7. Mr. Mantha Lakshman Murthi, M.A. "On the method of symbolic long division applied to the solutions of linear differential equations." [*Communicated by Mr. N. K. Majumdar, M.A.*]

8. Prof. S. K. Banerji, D.Sc. and Mr. Jyotirmaya Ghosh, M.A. "Preliminary report of an enquiry into the mathematical theories of earthquake waves."

9. Mr. Prabodhchandra Sengupta, M.A. "On Aryabhata's method of determining the mean motions of planets."

10. Mr. Surendramohan Sen, M.A. "Phoronomy in the Double-origin Angular Co-ordinates."

11. Mr. Nikhilranjan Sen, M.Sc. "On a Potential Problem."

12. Mr. L. Srivastava, M.Sc. "A note on the theory of magnetometer." [*Communicated by the Secretary.*]

13. Mr. Oudh Upadhyaya. "On some new Tautochrones."

14. Prof. C. E. Cullis, M.A., Ph.D. "The evaluation of the product matrix in a commutantal product of simple matrices having given nullities."

15. Prof. S. K. Banerji, D.Sc. "On the evaluation of the integral of the product of any number of Legendre functions of different degrees."

16. Mr. Nalinimohan Basu, M.Sc. "On liquid motion inside rotating arcs of three and four confocal parabolas."

17. Mr. Haripada Datta, M.A. "On the theory of Continued Fraction (third paper)."

18. Mr. Sasindrachandra Dhar, M.Sc. "On the convergence of the solution of Mathieu's Equation of the second kind."

19. Mr. J. C. Kameswar Rao, M.Sc. "Investigations on Ripple Motion." [*Communicated by Prof. C. V. Raman, M.A.*]

20. Mr. Jyotirmaya Ghosh, M.Sc. "Radial strain in a gravitating heterogeneous sphere."

21. Mr. Jyotirmaya Ghosh, M.Sc. "Plain strain in a thin elastic lamina."

22. Mr. Oudh Upadhyaya. "A note on Fermat's number."

23. Mr. Nripendranath Sen, M.Sc. "On vertically propagated atmospheric waves of finite amplitude."

24. Prof. M. Fréchet of the University of Strasbourg. "Sur divers modes de convergence d'une suite de fonctions d'une variable.' [*Communicated by the Secretary.*]

25. Prof. S. K. Banerji, D.Sc. "On the Problem of Submarine Signalling."

26. Mr. Haripada Datta, M.A. "On the theory of continued fraction (fourth paper)."

27. Mr. Abanibhushan Datta, M.A. "On an extension of Sonine's integral in Bessel functions."

28. Mr. Oudh Upadhyaya. "A note on quin-qui section."

29. Mr. Benode Bihari Paul, M.A. "Analytical Theory and Principle of Mathematical Evolution." [*Communicated by Mr. Devaprasad Ghosh, M.A.*]

30. Mr. Haripada Datta, M.A. "On some symmetric determinants."

31. Mr. Oudh Upadhyaya. "On Cyclotomic Hepta-section.

32. Mr. Dhirendranath Mookerjee. "On Libration of Equinoxes."

S. K. BANERJI, D.Sc.,
Secretary.

APPENDIX B.

Income and Expenditure Accounts, 1920.

1919. Rs.	A.	F.	INCOME.	1920. Rs.	A.	P.
322	13	0	Opening Balance ..	475	7	6
1,067	0	0	Admission Fee and Subscriptions—	911	0	0
			For 1917 ... 3 0 0			
			,, 1918 ... 51 0 0			
			,, 1919 ... 240 0 0			
			,, 1920 ... 617 0 0			
			Bulletin Sale Proceeds ...	80	12	9
			Reprint Charges realised ..	23	0	0
			Miscellaneous Receipts, being difference in the closing balance of 1919 ...	0	12	0
1,389	13	0		1,491	0	3

1919. Rs.	A.	P.	EXPENDITURE.	1920. Rs.	A.	P.
70	12	0	Books and Periodicals ...	74	1	0
5	8	0	Payment of out-standing bill of previous year ..	6	10	0
119	8	0	Binding Charges ...	66	0	0
			Bulletin Printing Charges	75	10	0
			Despatching Charges ...	20	2	0
289	13	6	Meeting Expenses, including Printing of notices, etc. ...	91	4	3
			Miscellaneous Charges ..	12	13	0
428	0	0	Establishment—	534	0	0
			Clerk ... 240 0 0			
			Typist .. 90 0 0			
			Bearer .. 204 0 0			
476	3	6	Closing Balance ...	610	8	0
1,389	13	0		1,491	0	3

We have great pleasure in reporting that we have audited the above account for 1920 of the Calcutta Mathematical Society with the books of the Society and the vouchers relating thereto, and we certify that it is correct.

N. K. MAJUMDAR, M.A.,
B. N. PAL, M.A,
Auditors.

Calcutta, the 28th January. 1921.